MATH102

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5.2: Area

Objectives

> “
>
> 1. Use sigma notation to write and evaluate a sum.
> 2. Understand the concept of area.
> 3. Approximate the area of a plane region.
> 4. Find the area of a plane region using limits.

Sigma Notation

Sigma Notation

The sum of n terms $a_1, a_2, \cdots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

where i is the **index of summation**, a_i is the th **ith term** of the sum, and the upper and lower bounds of summation are n and 1.

Summation Properties

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Theorem **Summation Formulas**

$$(1)\quad \sum_{i=1}^{n} c = cn, \quad c \text{ is a constant}$$

$$(2)\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(3)\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(4)\quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1: Evaluating a Sum

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n}$ for $n = 10, 100, 1000$ and $10,000$.

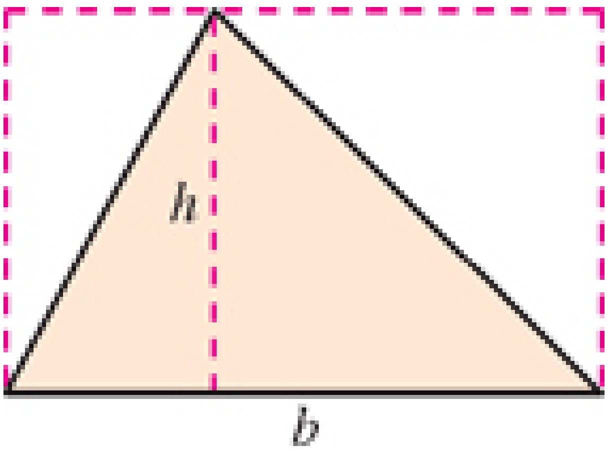
Area

In **Euclidean geometry**, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is

$$A = bh$$

it is actually more proper to say that this is the *definition* of the **area of a rectangle**.

For a triangle $A = \frac{1}{2}bh$



The Area of a Plane Region

Example

Use **five** rectangles to find two approximations of the area of the region lying between the graph of

$$f(x) = 5 - x^2$$

and the x-axis between $x = 0$ and $x = 2$.

```
f (generic function with 1 method)
```

```
f(x) = 5 - x^2
```

n = 5 a = 0 b = 2 method = Left

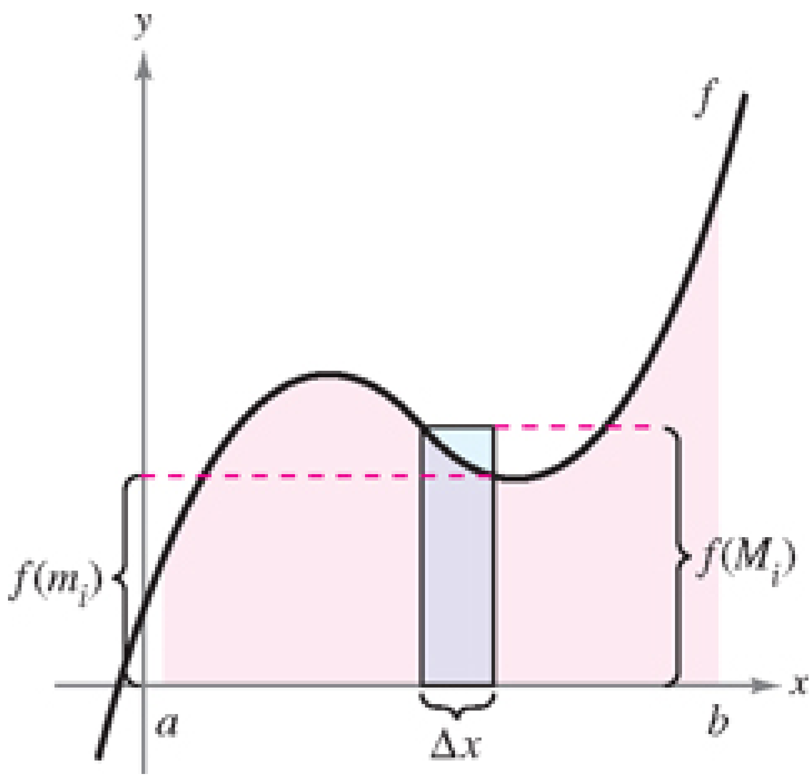
○ ✓
◉ ✗

Finding Area by the Limit Definition

Find the area of a plane region bounded above by the graph of a nonnegative, **continuous** function

$$y = f(x)$$

The region is bounded below by the x- axis and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$



- To approximate the area of the region, begin by subdividing the interval into subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

- The endpoints of the intervals are

$$\overbrace{a + 0(\Delta x)}^{a=x_0} < \overbrace{a + 1(\Delta x)}^{a=x_1} < \overbrace{a + 2(\Delta x)}^{a=x_2} < \cdots < \overbrace{a + n(\Delta x)}^{a=x_n}.$$

- Let

$$\begin{aligned} f(m_i) &= \text{Minimum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \\ f(M_i) &= \text{Maximum value of } f(x) \text{ on the } i^{\text{th}} \text{ subinterval} \end{aligned}$$

- Define an **inscribed rectangle** lying inside the i^{th} subregion
- Define an **circumscribed rectangle** lying outside the i^{th} subregion

$$(\text{Area of inscribed rectangle}) = f(m_i)\Delta x \le f(M_i)\Delta x = (\text{Area of circumscribed rectangle})$$

- The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of the circumscribed rectangles is called an **upper sum**.

$$\begin{aligned} \text{Lower sum} &= s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \quad \text{Area of inscribed rectangle} \\ \text{Upper sum} &= S(n) = \sum_{i=1}^{n} f(M_i)\Delta x \quad \text{Area of circumscribed rectangle} \end{aligned}$$

- The actual area of the region lies between these two sums.

$$s(n) \le (\text{Area of region}) \le S(n).$$

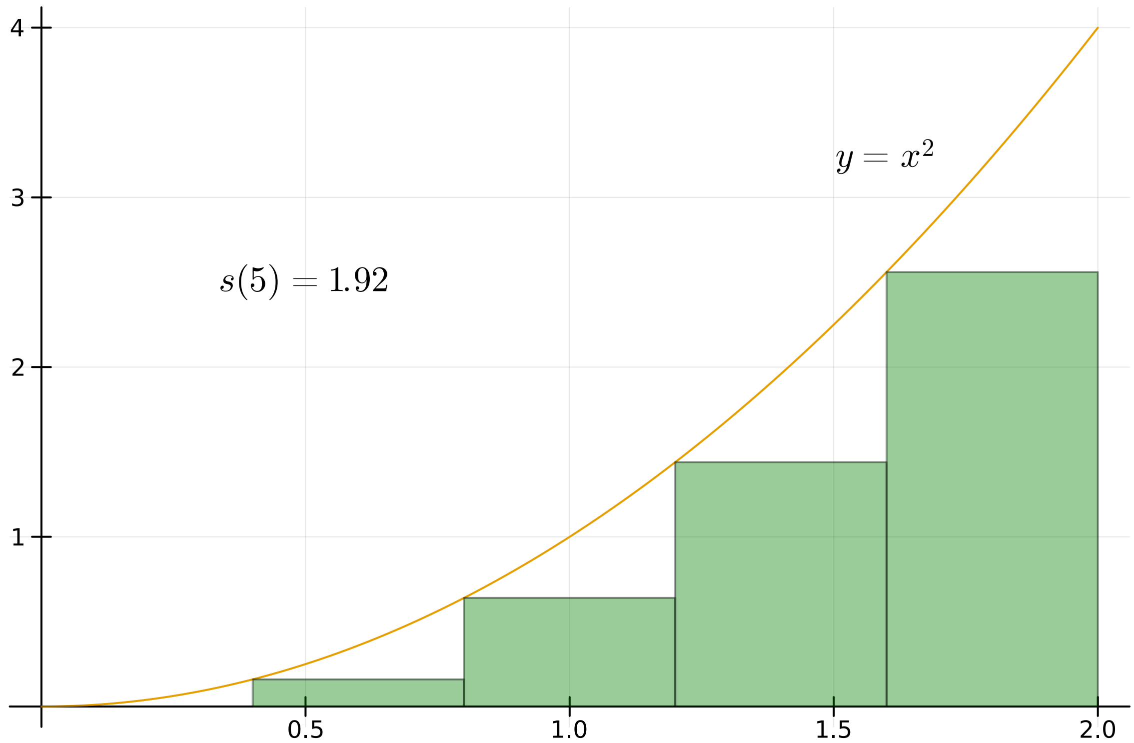
Example 4: Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the $x-$axis between $x = 0$ and $x = 2$.

n = 5 a = 0 b = 2 method = Left

```
f4 (generic function with 1 method)
```

```
f4(x) = x^2
```



Theorem **Limits of the Lower and Upper Sums**

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

where

$$\Delta x = \frac{b-a}{n}$$

and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the ith subinterval.

Definition **Area of a Region in the Plane**

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $y = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where

$$x_{i-1} \leq c_i \leq x_i \quad \text{and} \quad \Delta x = \frac{b-a}{n}.$$

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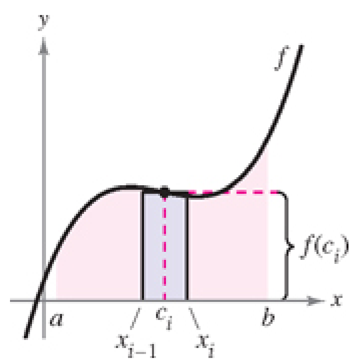


Example 5: Finding Area by the Limit Definition

Find the area of the region bounded by the graph of $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Example 7: A Region Bounded by the *y*-axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \leq y \leq 1$.))

Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

Example 8: **Approximating Area with the Midpoint Rule**

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = \sin x$ and the x-axis for $0 \le x \le \pi$.

```
2.0523443059540623
```

```
begin
    f8(x)=sin(x)
    Δx28 = π/4
    A = Δx28*(f8(π/8)+f8(3π/8)+f8(5π/8)+f8(7π/8))
end
```

5.3: Riemann Sums and Definite Integrals

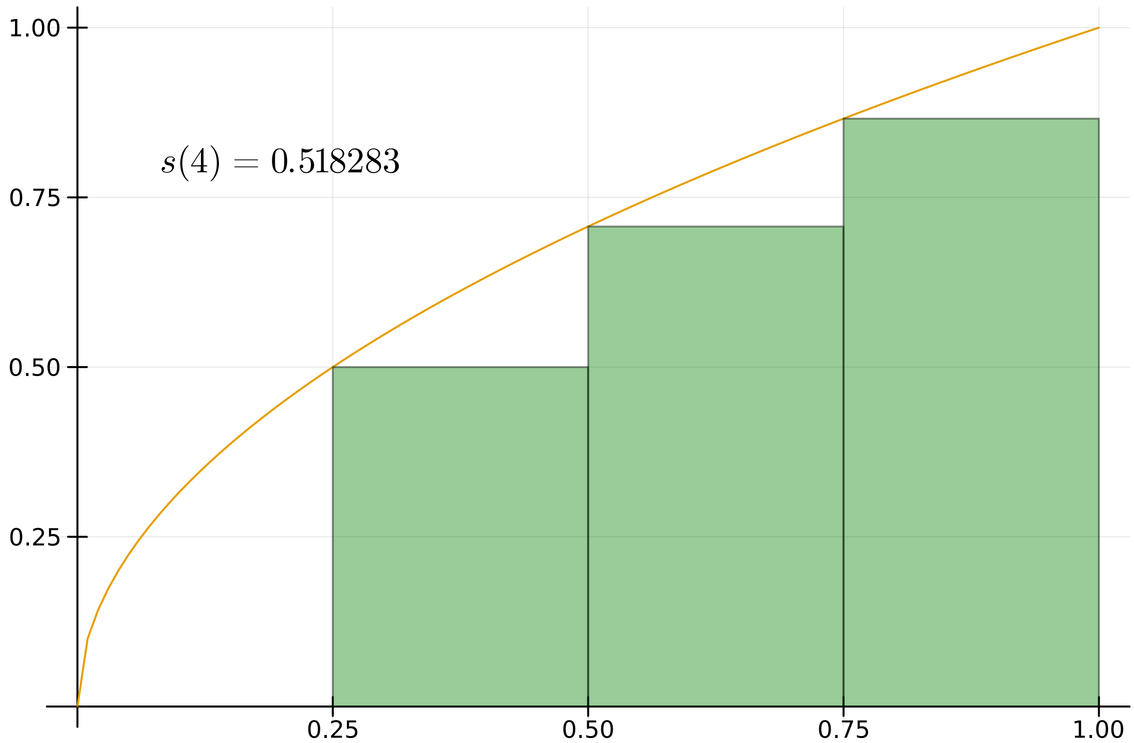
> **Objectives**
>
> 1. Understand the definition of a Riemann sum.
> 2. Evaluate a definite integral using limits and geometric formulas.
> 3. Evaluate a definite integral using properties of definite integrals.

Riemann Sums

```
g (generic function with 1 method)
```

```
g(x) = √x
```

n = 4 a = 0 b = 1 method = Left



Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the th subinterval

$$[x_{i-1}, x_i] \quad \textcolor{red}{i\text{th subinterval}}$$

If c_i is any point in the th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

Remark

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$.

- If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n} \quad \text{Regular partition}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \quad \text{General partition}$$

- Note that

$$\|\Delta\| \to 0 \quad \text{implies that} \quad n \to \infty.$$

Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists, then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Theorem **Continuity Implies Integrability**

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is,

$$\int_a^b f(x)dx \quad \text{exists.}$$

Theorem **The Definite Integral as the Area of a Region**

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)dx$$

Example 3: Areas of Common Geometric Figures

Evaluate each integral using a geometric formula.

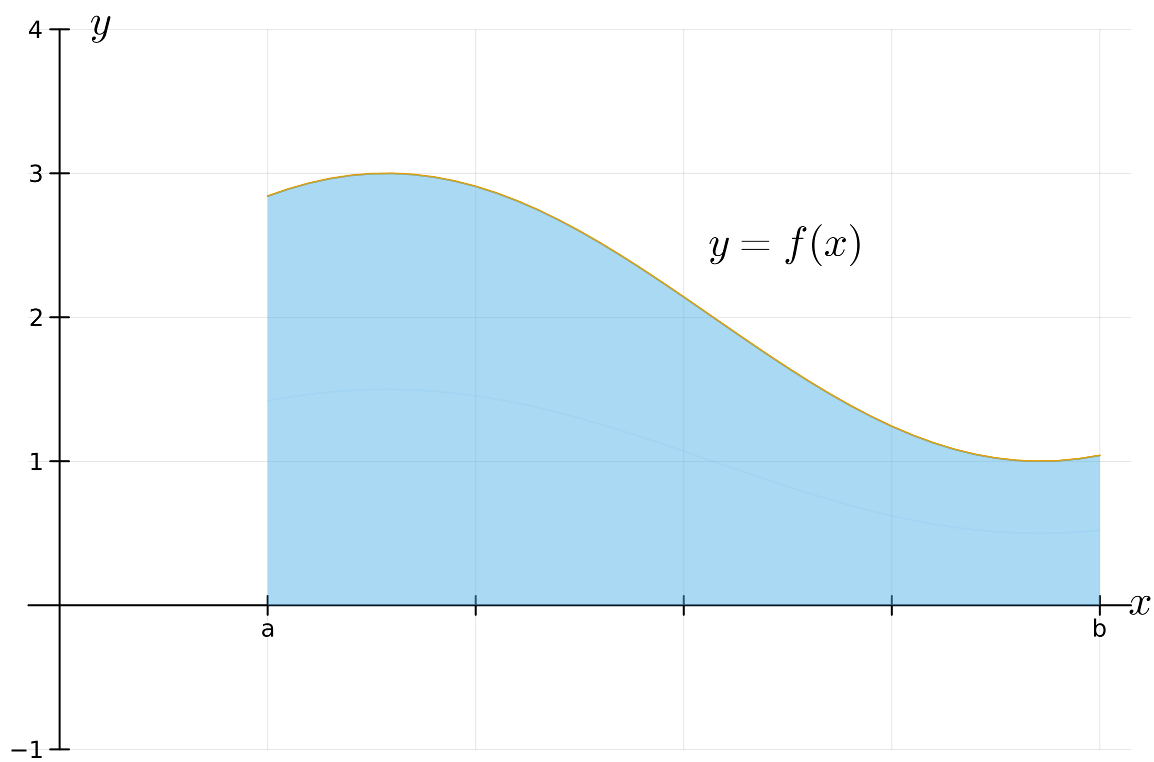
- $\int_1^3 4dx$
- $\int_0^3 (x+2)dx$
- $\int_{-2}^2 \sqrt{4-x^2}dx$

Remark **The definite integral is a **number****

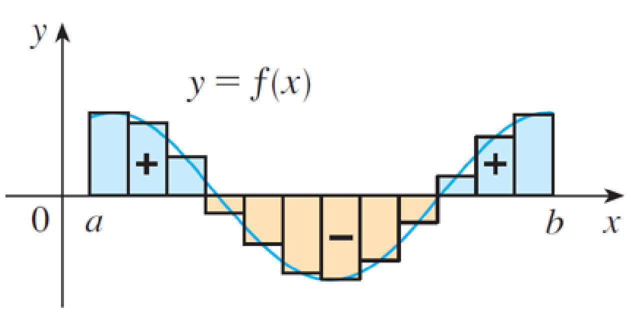
- It does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

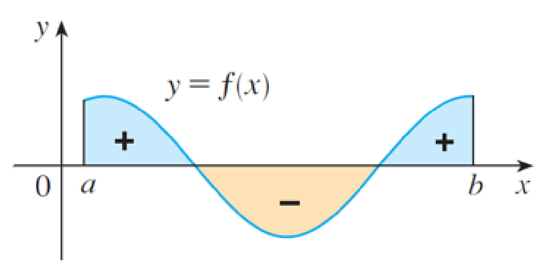
$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(w)dw = \int_a^b f(\text{😀})d\text{😀}$$

- If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from a to b.



- $\int_a^b f(x)dx$ is the net area





Properties of Definite Integrals

Definitions **Two Special Definite Integrals**

- If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$.
- If f is integrable on $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Theorem **Additive Interval Property**

If f is integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem **Properties of Definite Integrals**

- If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and
  1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
  2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

Theorem **Preservation of Inequality**

- If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)dx.$$

- If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

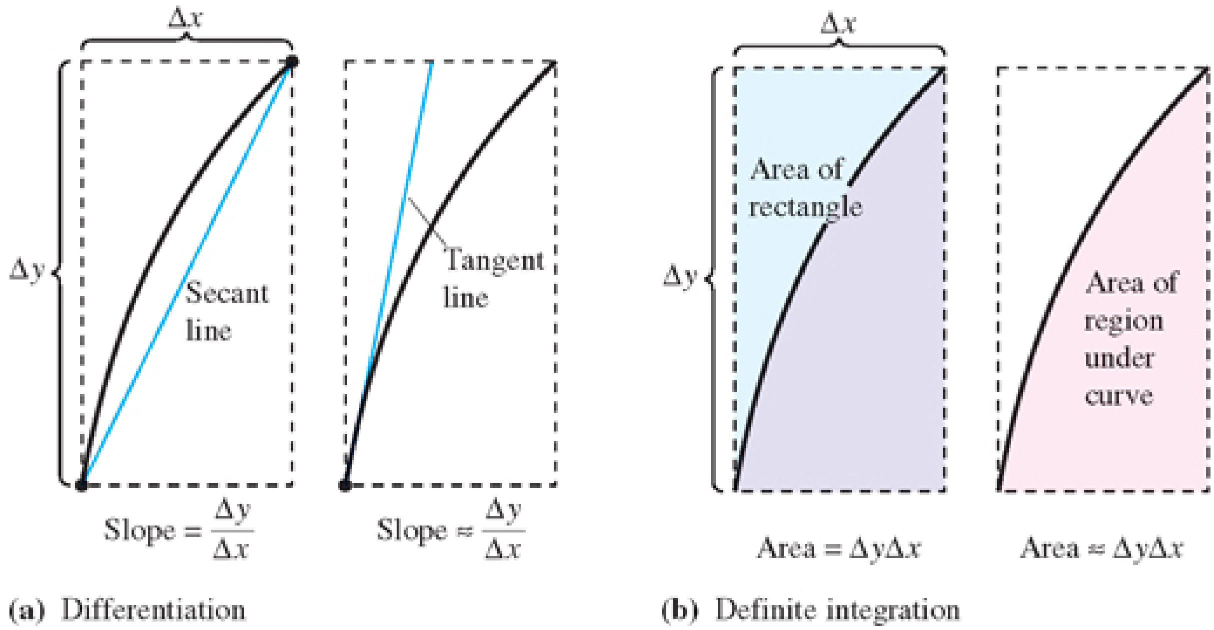
Examples:

5.4: The Fundamental Theorem of Calculus

> **Objectives**
>
> 1. Evaluate a definite integral using the Fundamental Theorem of Calculus.
> 2. Understand and use the Mean Value Theorem for Integrals.
> 3. Find the average value of a function over a closed interval.
> 4. Understand and use the Second Fundamental Theorem of Calculus.
> 5. Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus

Antidifferentiation and Definite Integration



- ✒ $\int_a^b f(x)dx$
  - definite integral
  - number
- ✒ $\int f(x)dx$
  - indefinite integral
  - function

Theorem **The Fundamental Theorem of Calculus**

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark

We use the notation

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{or} \quad \int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example 1: Evaluating a Definite Integral

Evaluate each definite integral.

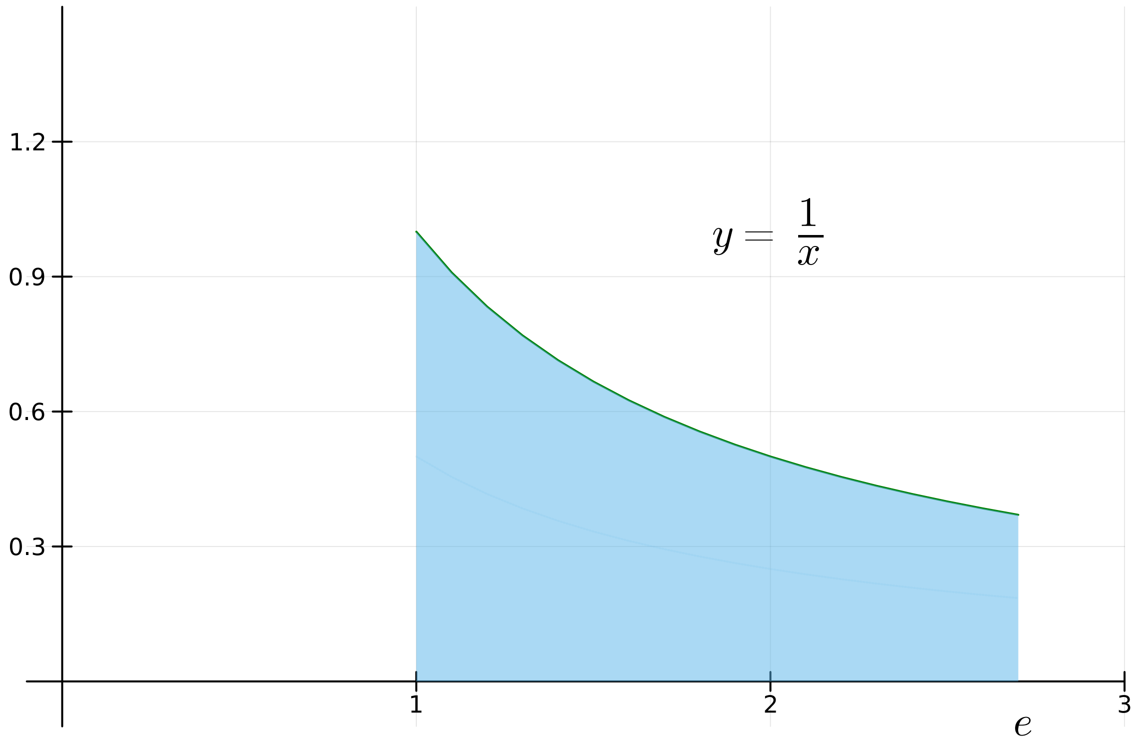
- $\int_1^2 (x^2 - 3)dx$
- $\int_1^4 3\sqrt{x}dx$
- $\int_0^{\pi/4} \sec^2 x dx$
- $\int_0^2 \Big|2x - 1\Big|dx$

Example 3: **Using the Fundamental Theorem to Find Area**

Find the area of the region bounded by the graph of

$$y = \frac{1}{x}$$

the x-axis, and the vertical lines $x = 1$ and $x = e$.

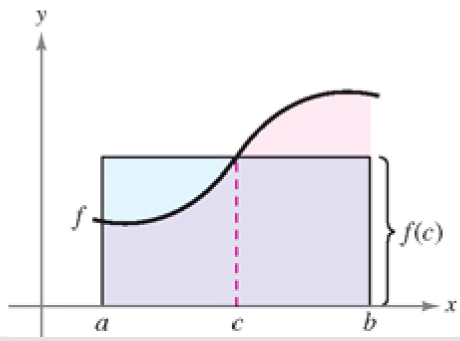


The Mean Value Theorem for Integrals

Theorem **The Mean Value Theorem for Integrals**

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$



Average Value of a Function

Definition **the Average Value of a Function on an Interval**

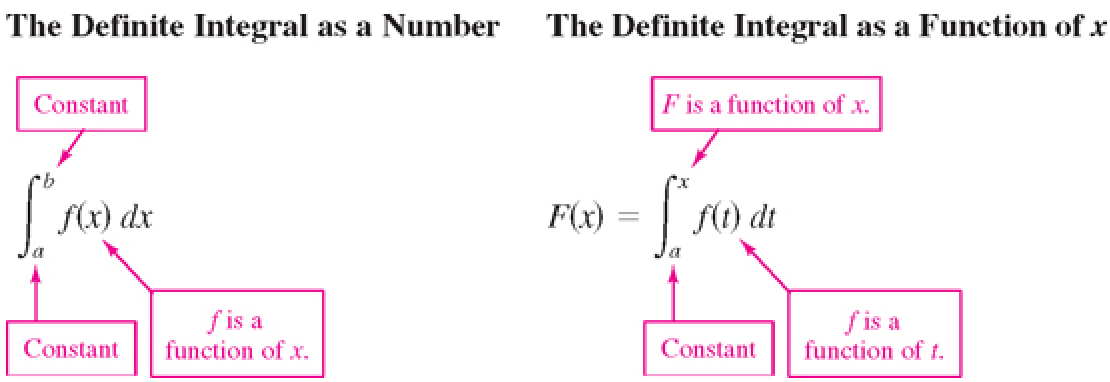
If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\textbf{Avergae value} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 4: **Finding the Average Value of a Function**

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

The Second Fundamental Theorem of Calculus

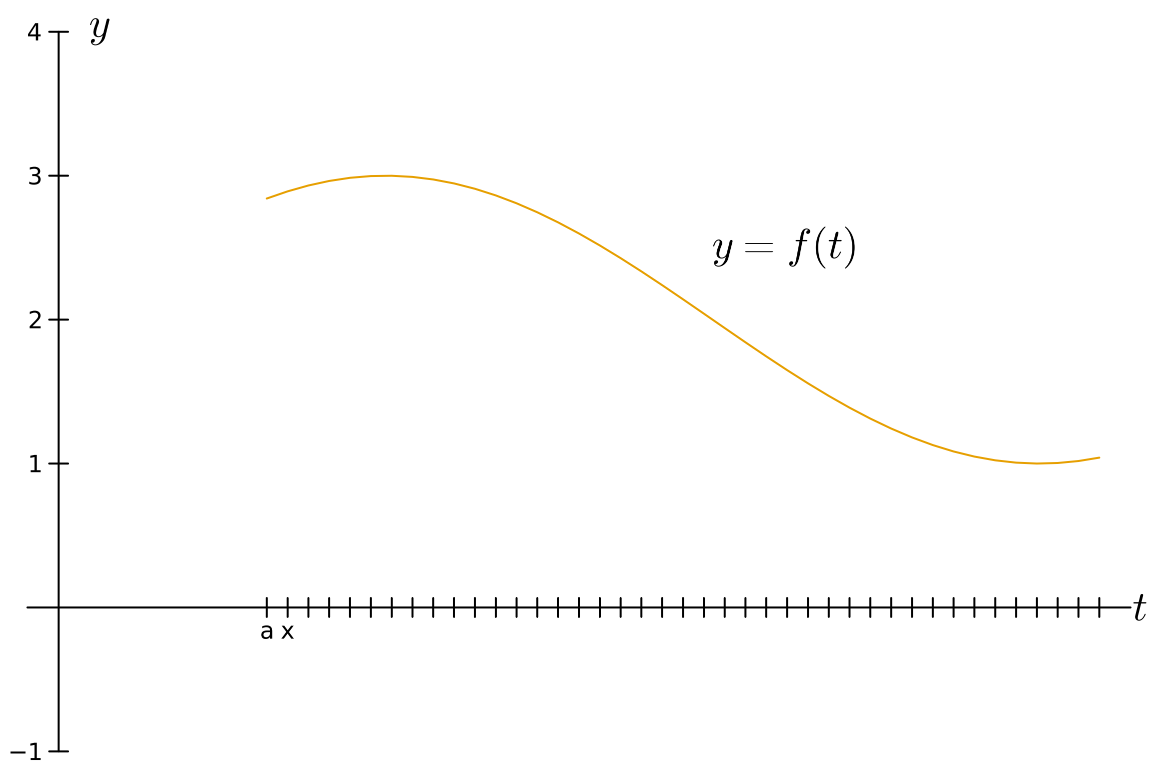


Consider the following function

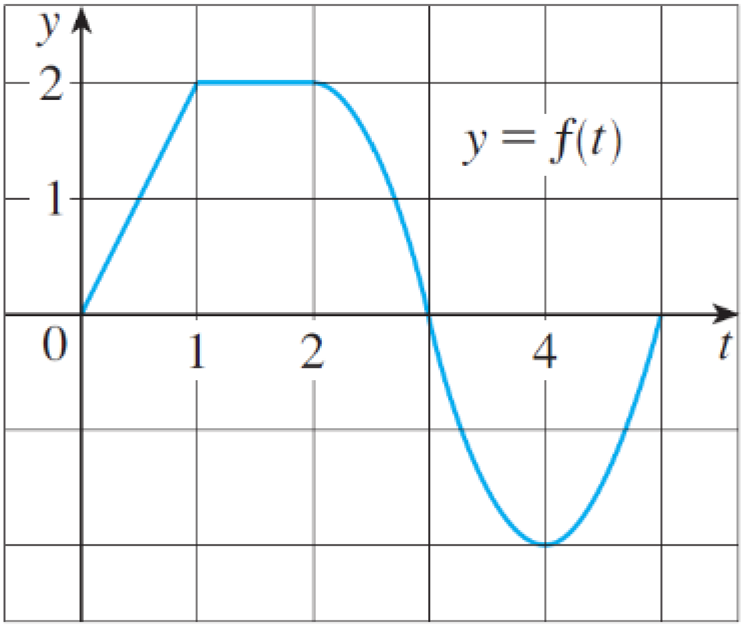
$$F(x) = \int_a^x f(t)dt$$

where f is a continuous function on the interval $[a, b]$ and $x \in [a, b]$.

x =



Example If $g(x) = \int_0^x f(t)dt$



Find $g(2)$

Theorem **The Second Fundamental Theorem of Calculus**

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\right] = f(x).$$

Remarks

- $\frac{d}{dx}\left(\int_a^x f(u)du\right) = f(x)$
- $g(x)$ is an **antiderivative** of f

Examples

Find the derivative of

(1) $g_1(x) = \int_0^x \sqrt{1+t}dt$.

(2) $g_2(x) = \int_x^0 \sqrt{1+t}dt$.

(3) $g_3(x) = \int_0^{x^2} \sqrt{1+t}dt$.

(4) $g_4(x) = \int_{\sin(x)}^{\cos(x)} \sqrt{1+t}dt$.

💣 BE CAREFUL:

Evaluate $\int_{-3}^{6} \frac{1}{x}dx$

Net Change Theorem

Question: If $y = F(x)$, then what does $F'(x)$ represents?

Theorem **The Net Change Theorem**

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)dx = F(b) - F(a) \qquad \text{Net change of } F(x)$$

- There are many applications, we will focus on one

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

- **Remarks**

$$\text{displacement} = \int_{t_1}^{t_2} v(t)dt$$

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)|dt$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1) \quad \text{is the change in velocity from time to time .}$$

Example 10: Solving a Particle Motion Problem

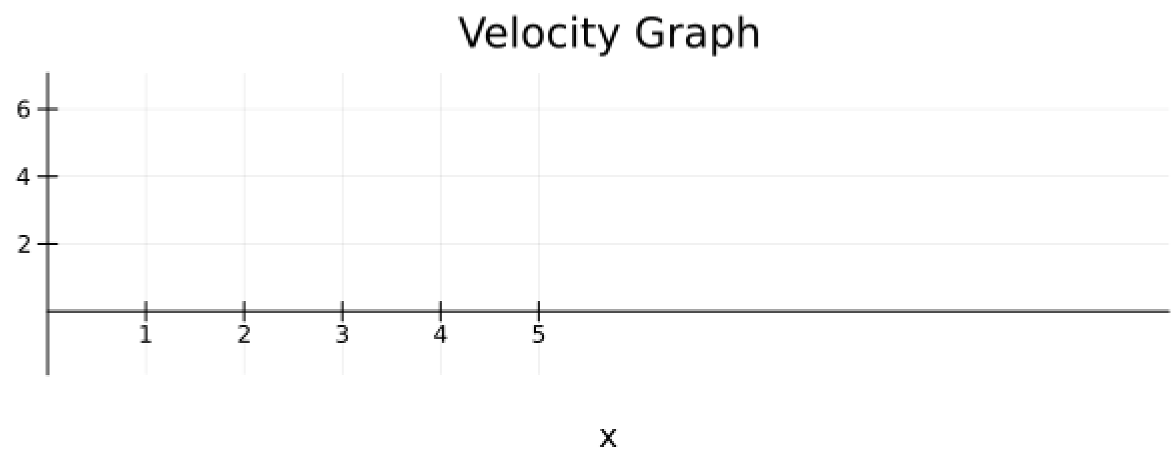
A particle is moving along aline. Its velocity function (in m/s^2) is given by

$$v(t) = t^3 - 10t^2 + 29t - 20,$$

a. What is the **displacement** of the particle on the time interval 1≤ t≤ 5?
b. What is the **total distance** traveled by the particle on the time interval 1≤ t≤ 5?

```
v (generic function with 1 method)
```

```julia
v(t) = t^3 - 10 * t^2 + 29 * t - 20
```

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5.5: The Substitution Rule

> **Objectives**
>
> 1. Use pattern recognition to find an indefinite integral.
> 2. Use a change of variables to find an indefinite integral.
> 3. Use the General Power Rule for Integration to find an indefinite integral.
> 4. Use a change of variables to evaluate a definite integral.
> 5. Evaluate a definite integral involving an even or odd function.

	solve	
$\int 2x\sqrt{1+x^2}\ \ dx$		$\int \sqrt{u}\ \ du$

Pattern Recognition

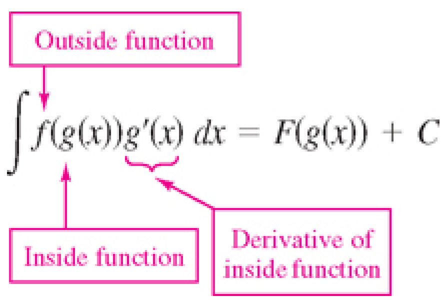
Theorem **Antidifferentiation of a Composite Function**

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C.$$



Substitution Rule says: It is permissible to operate with dx and du after integral signs as if they were differentials.

Example Find

$$(i) \qquad \int \left(x^2+1\right)^2 (2x)dx$$

$$(ii) \qquad \int 5e^{5x}dx$$

$$(iii) \qquad \int \frac{x}{\sqrt{1-4x^2}}dx$$

$$(iv) \qquad \int \sqrt{1+x^2} \;\; x^5 dx$$

$$(v) \qquad \int \tan x dx$$

Change of Variables for Indefinite Integrals

Example: Find

$$(i) \qquad \int \sqrt{2x-1}dx$$

$$(ii) \qquad \int x\sqrt{2x-1}dx$$

$$(iii) \quad \int \sin^2 3x \cos 3x dx$$

The General Power Rule for Integration

Theorem **The General Power Rule for Integration**

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example: Find

$$(i) \quad \int 3(3x-1)^4 dx$$

$$(ii) \quad \int (e^x+1)(e^x+x)dx$$

$$(iii) \quad \int 3x^2\sqrt{x^3-2}\, dx$$

$$(iv) \quad \int \frac{-4x}{(1-2x^2)^2}\, dx$$

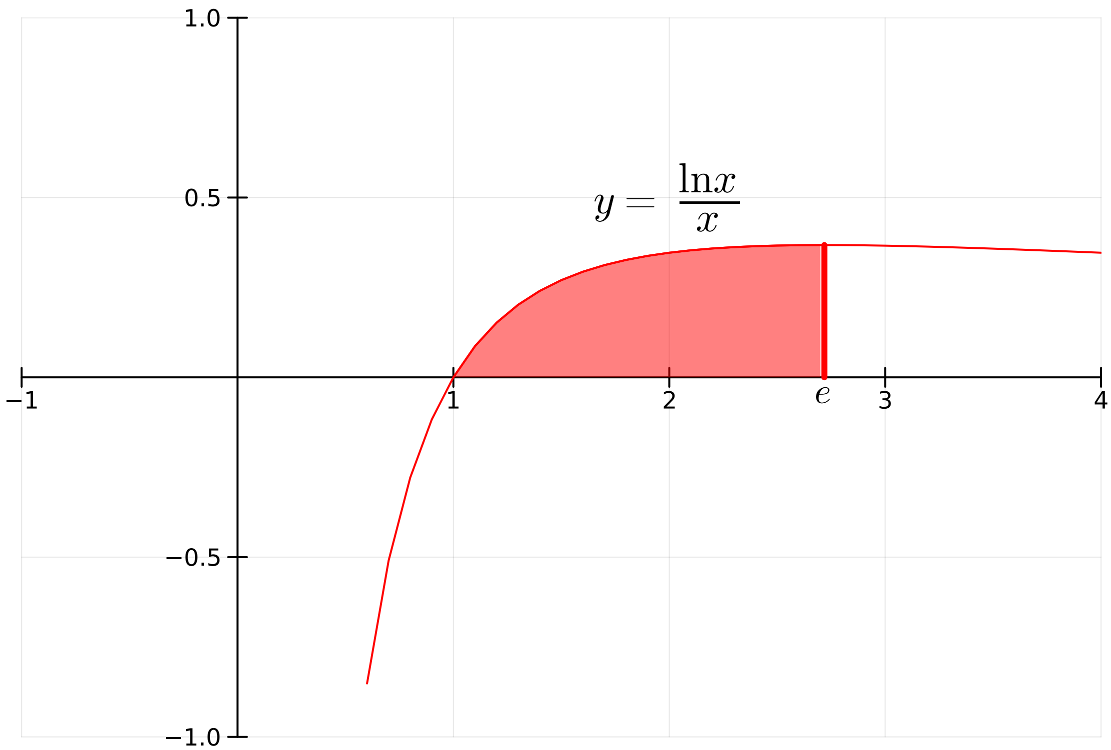
$$(v) \quad \int \cos^2 x \sin x \; dx$$

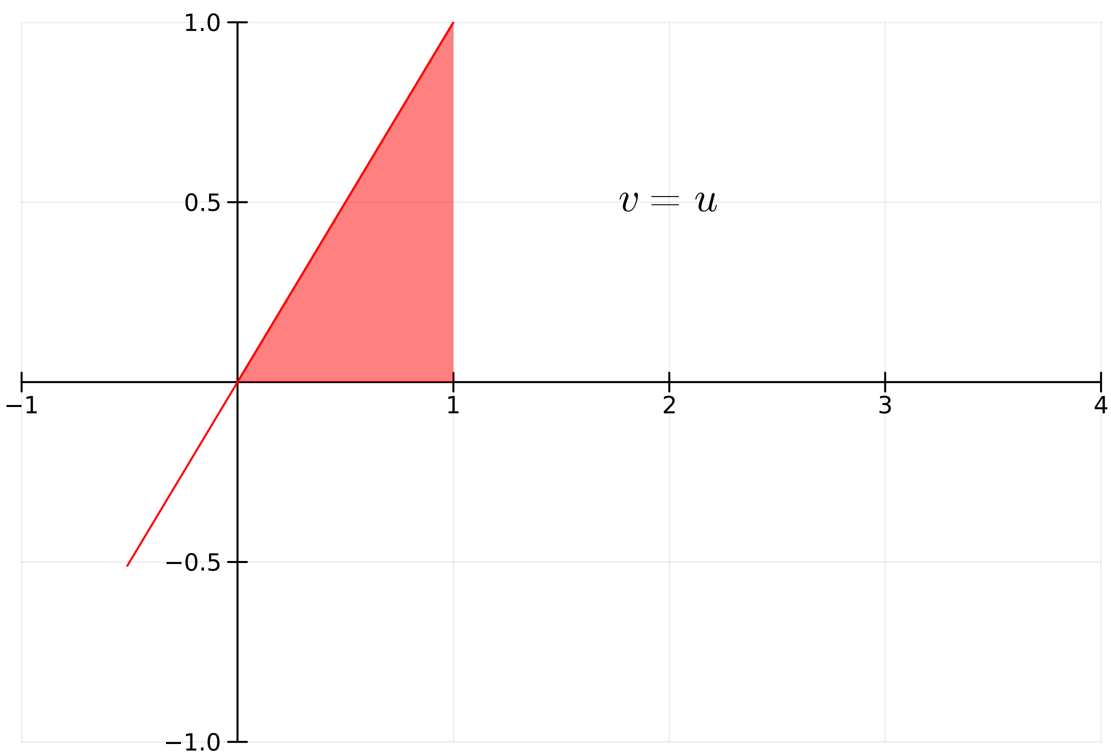
Change of Variables for Definite Integrals

Substitution: Definite Integrals

Example: Evaluate

$$\int_1^e \frac{\ln x}{x} dx$$





Example: Evaluate

$$(i) \qquad \int_1^2 \frac{dx}{(3-5x)^2}$$

$$(iii) \quad \int_0^1 x(x^2+1)^3 \, dx$$

$$(iv) \quad \int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$$

Integration of Even and Odd Functions

Theorem **Integration of Even and Odd Functions**

Let f be integrable on $[-a, a]$.

- If f is **even** $[f(-x) = f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

- If f is **odd** $[f(-x) = -f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

Example Find

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx$$

5.7: The Natural Logarithmic Function: Integration

> **Objectives**
>
> 1. Use the Log Rule for Integration to integrate a rational function.
> 2. Integrate trigonometric functions.

Log Rule for Integration

Theorem **Log Rule for Integration**

Let u be a differentiable function of x.

$$\begin{aligned} &\text{(i)} \quad \int \frac{1}{x} dx &= \ln|x| + C \\ &\text{(ii)} \quad \int \frac{1}{u} du &= \ln|u| + C \end{aligned}$$

Remark

$$\int \frac{u'}{u} dx = \ln|u| + C$$

Example 1: Using the Log Rule for Integration

$$\int \frac{2}{x} dx$$

Example 3: Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.

Example 5: Using Long Division Before Integrating

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx$$

Examples Find

$$\text{(i)} \quad \int \frac{1}{4x-1} dx$$

$$\text{(ii)} \quad \int \frac{3x^2+1}{x^3+x} dx$$

$$\text{(iii)} \quad \int \frac{\sec^2 x}{\tan x} dx$$

$$\text{(iv)} \quad \int \frac{x^2+x+1}{x^2+1} dx$$

$$\text{(v)} \quad \int \frac{2x}{(x+1)^2} dx$$

Example 7: **Solve the differential equation**

Solve

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

Integrals of Trigonometric Functions

Example 8: **Using a Trigonometric Identity**

$$\int \tan x dx$$

Example 9: **Derivation of the Secant Formula**

$$\int \sec x dx$$

5.8:Inverse Trigonometric Functions: Integration

> **Objectives**
>
> 1. Integrate functions whose antiderivatives involve inverse trigonometric functions.
> 2. Use the method of completing the square to integrate a function.
> 3. Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

Theorem **Integrals Involving Inverse Trigonometric Functions**

Let u be a differential function of x, and let $a > 0$.

$$
\begin{aligned}
1.\quad & \int \frac{du}{\sqrt{a^2-u^2}} &=& \arcsin\tfrac{u}{a} + C \\
2.\quad & \int \frac{du}{a^2+u^2} &=& \tfrac{1}{a}\arctan\tfrac{u}{a} + C \\
3.\quad & \int \frac{du}{u\sqrt{u^2-a^2}} &=& \tfrac{1}{a}\operatorname{arcsec}\tfrac{|u|}{a} + C
\end{aligned}
$$

Examples Find

- $$\int \frac{dx}{\sqrt{4-x^2}},$$
- $$\int \frac{dx}{2+9x^2},$$
- $$\int \frac{dx}{x\sqrt{4x^2-9}},$$
- $$\int \frac{dx}{\sqrt{e^{2x}-1}},$$
- $$\int \frac{x+2}{\sqrt{4-x^2}}dx.$$

Completing the Square

Example 5: Completing the Square

Find

$$\int \frac{dx}{x^2 - 4x + 7}.$$

Example 6: Completing the Square

Find the area of the region bounded by the graph of

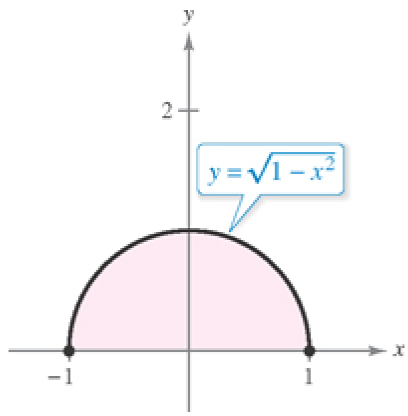
$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

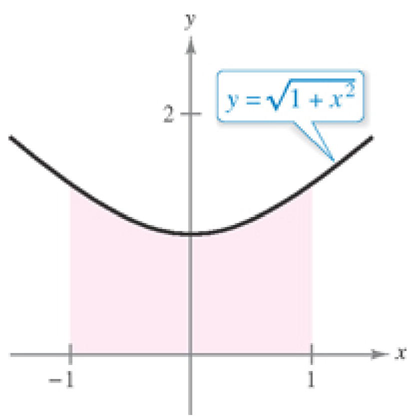
5.9: Hyperbolic Functions

> **Objectives**
>
> 1. Develop properties of hyperbolic functions (MATH101).
> 2. Differentiate (MATH101) and integrate hyperbolic functions.
> 3. Develop properties of inverse hyperbolic functions (Reading only).
> 4. Differentiate and integrate functions involving inverse hyperbolic functions. (Reading only).

Circle: $x^2 + y^2 = 1$



Hyperbola: $-x^2 + y^2 = 1$

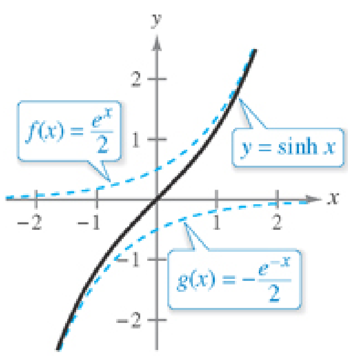


Definitions of the Hyperbolic Functions

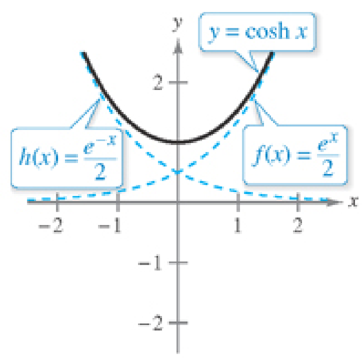
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x},\ x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

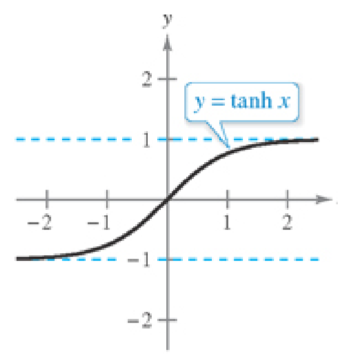
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x},\ x \neq 0$$



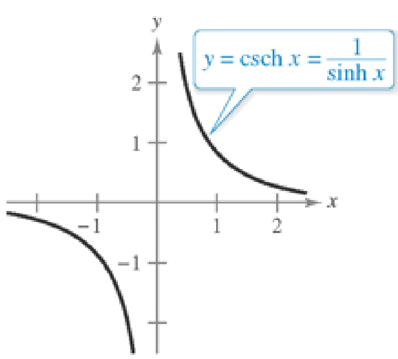
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



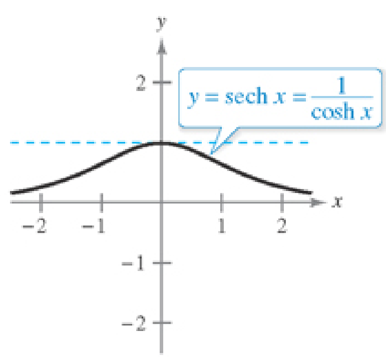
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



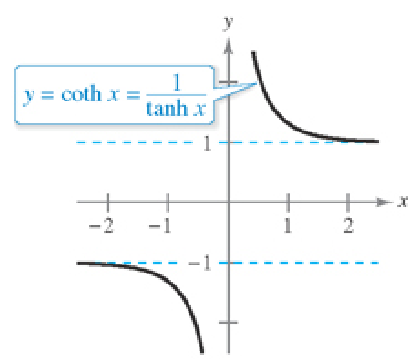
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Hyperbolic Identities

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1, & \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh \\
\tanh^2 x + \text{sech}^2 x &= 1, & \sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh \\
\coth^2 x - \text{csch}^2 x &= 1, & \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh \\
& & \cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh
\end{aligned}
$$

$$
\begin{aligned}
\sinh^2 x &= \frac{\cosh 2x - 1}{2}, & \cosh^2 x &= \frac{\cosh 2x + 1}{2} \\
\sin 2x &= 2 \sinh x \cosh x, & \cosh 2x &= \cosh^2 x + \sinh^2 x
\end{aligned}
$$

Theorem **Differentiation and Integration of Hyperbolic Functions**

Theorem Let u be a differentiable function of x.

$$\frac{d}{dx}(\sinh u) = (\cosh u)u', \qquad \int \cosh u du = \sinh u + C$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)u', \qquad \int \sinh u du = \cosh u + C$$

$$\frac{d}{dx}(\tanh u) = \left(\operatorname{sech}^2 u\right)u', \qquad \int \operatorname{sech}^2 u du = \tanh u + C$$

$$\frac{d}{dx}(\coth u) = -\left(\operatorname{csch}^2 u\right)u', \qquad \int \operatorname{csch}^2 u du = -\coth u + C$$

$$\frac{d}{dx}(\operatorname{sech} u) = -(\operatorname{sech} u \tanh u)u', \qquad \int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}(\operatorname{csch} u) = -(\operatorname{csch} u \coth u)u', \qquad \int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$$

Example 4: Integrating a Hyperbolic Function

Find

$$\int \cosh 2x \sinh^2 2x dx$$

7.1: Area of a Region Between Two Curves

Objectives

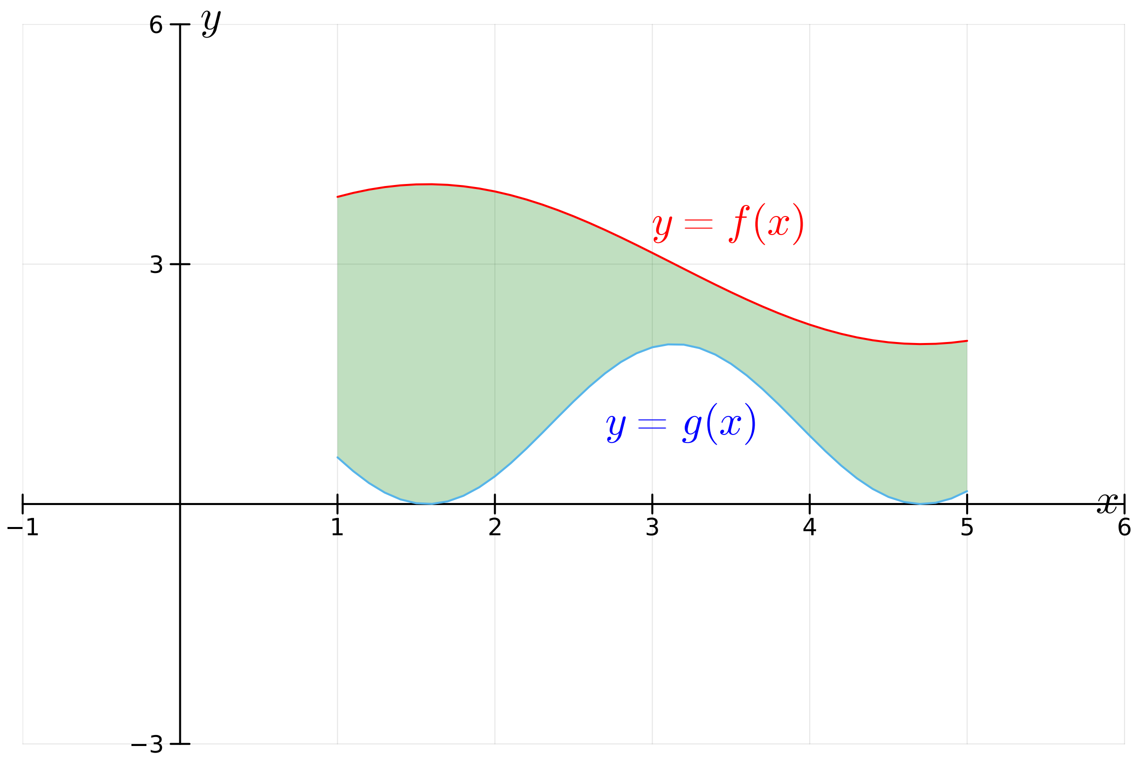
> “
>
> 1. Find the area of a region between two curves using integration.
> 2. Find the area of a region between intersecting curves using integration.
> 3. Describe integration as an accumulation process.

......

Area of a Region Between Two Curves

move n = 1

How can we find the area between the two curves?



$$Area = \int_a^b [f(x) - g(x)]dx$$

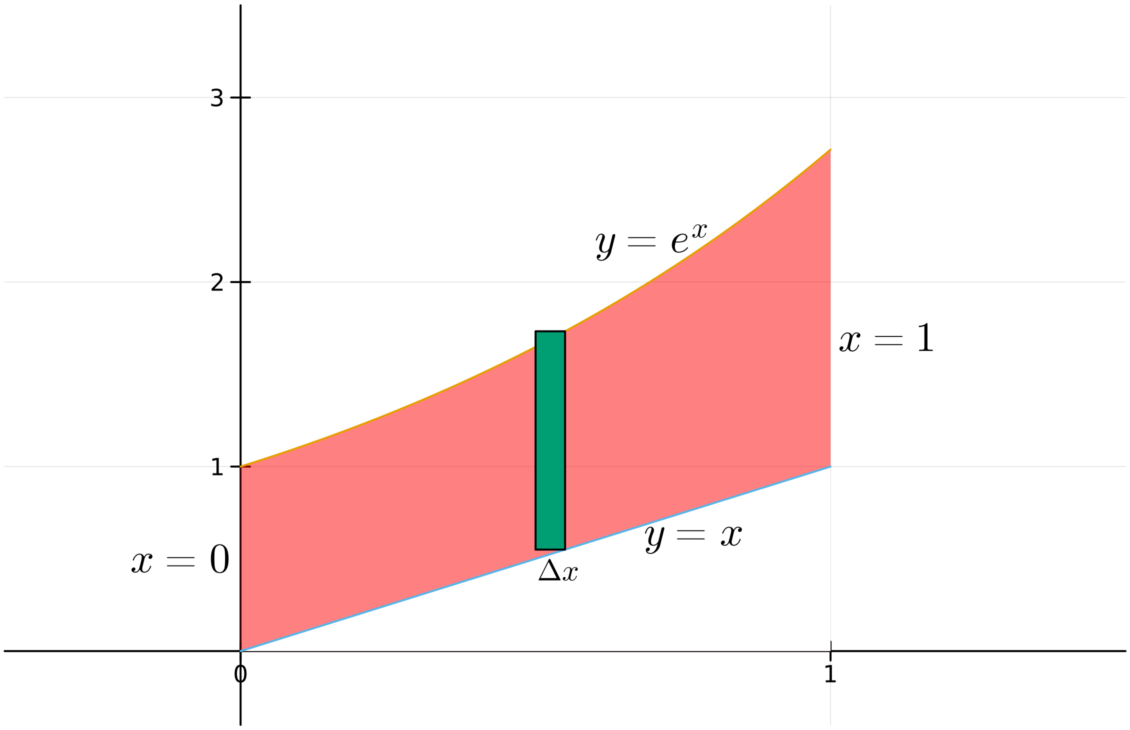
Remark

- Area = $y_{top} - y_{bottom}$.

Example 1: Finding the Area of a Region Between Two Curves

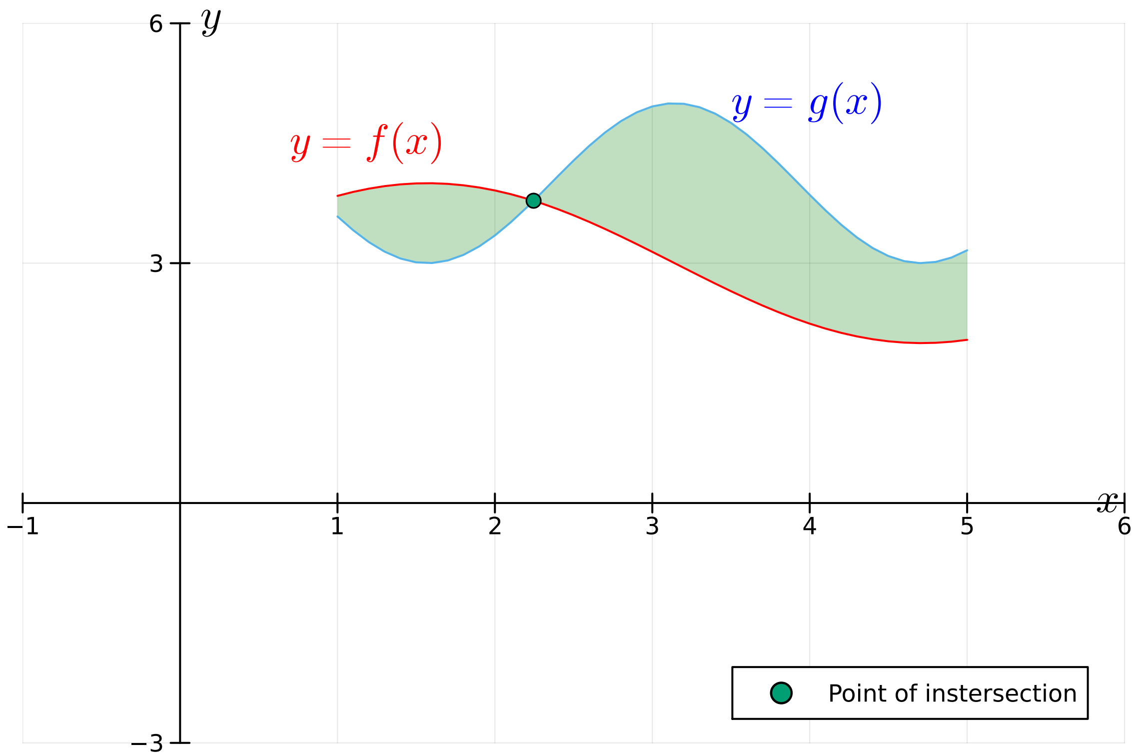
Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, bounded on the sides by $x = 0$ and $x = 1$.

Solution



Area of a Region Between Intersecting Curves

In geberal,



$$Area = \int_a^b |f(x) - g(x)| dx$$

Example 2: **A Region Lying Between Two Intersecting Graphs**

Find the area of the region enclosed by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution in class

Example 3: A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the curves

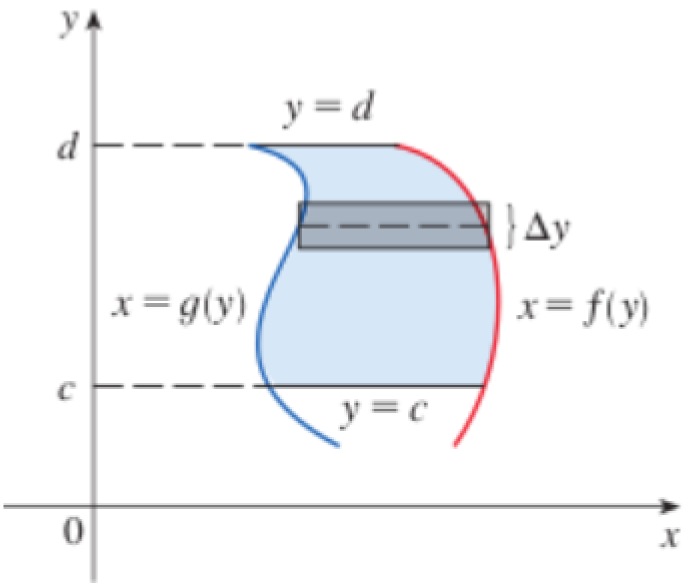
$$y = \cos(x), \quad y = \sin(x), \quad x = 0, \quad x = \frac{\pi}{2}$$

Example 4: Curves That Intersect at More than Two Points

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x, \qquad g(x) = -x^2 + 2x.$$

Integrating with Respect to y



Example 5: Horizontal Representative Rectangles

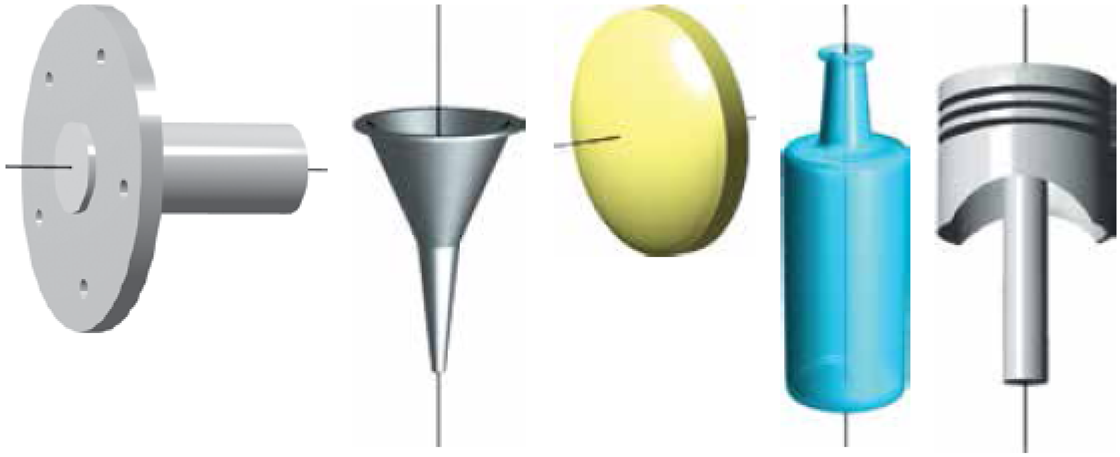
Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

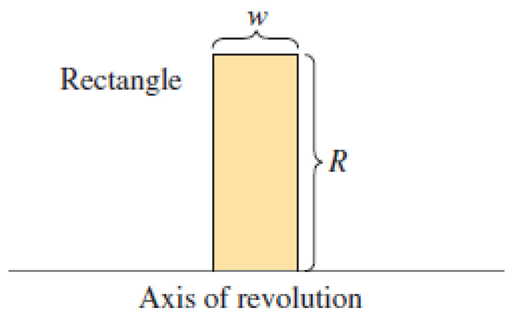
7.2: Volume: The Disk Method

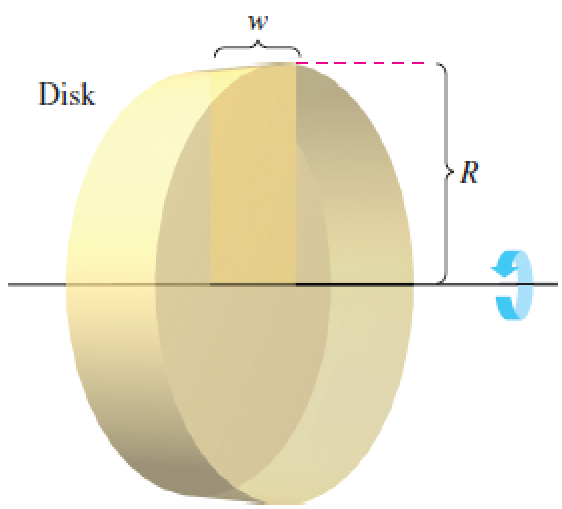
> **Objectives**
>
> - Find the volume of a solid of revolution using the disk method.
> - Find the volume of a solid of revolution using the washer method.
> - Find the volume of a solid with known cross sections.

The Disk Method

Solids of Revolution



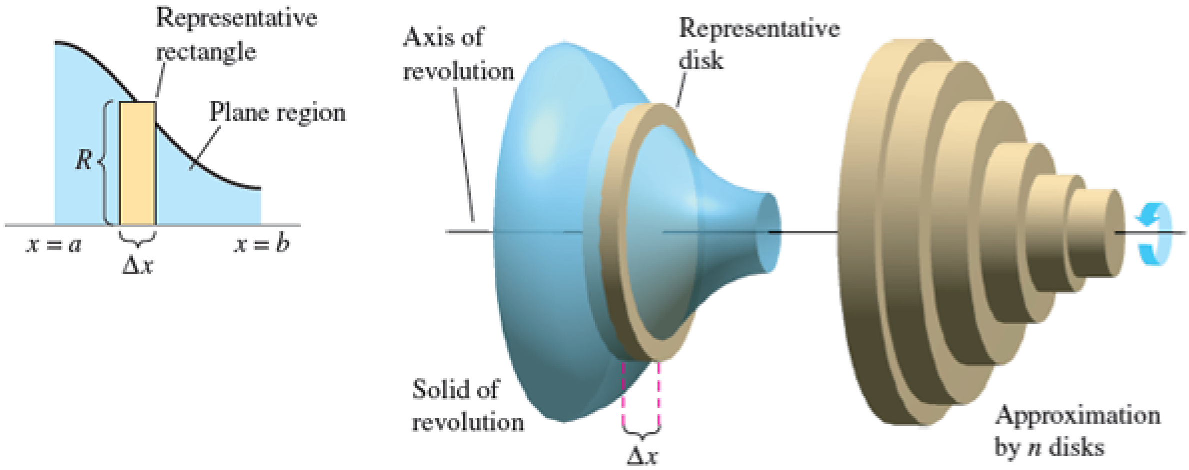




Volume of a disk

$$V = \pi R^2 w$$

Disk Method



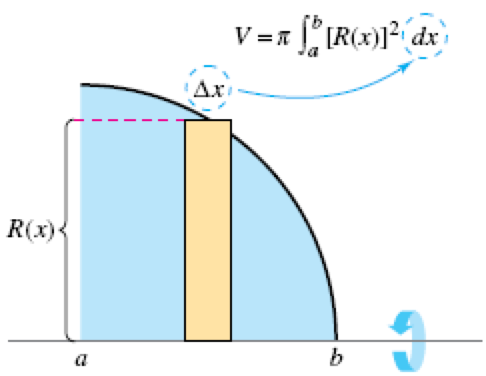
$$\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x$$
$$= \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x$$

Taking the limit $\|\Delta\| \to 0 (n \to \infty)$, we get

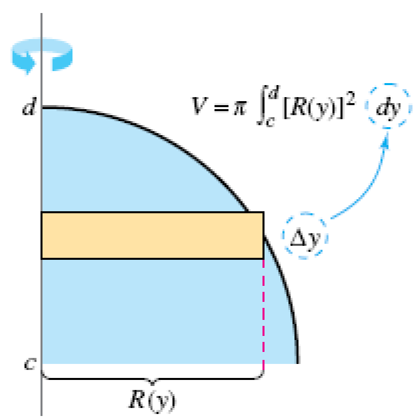
$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 dx.$$

Disk Method

To find the volume of a solid of revolution with the disk method, use one of the formulas below



Horizontal axis of revolution



Vertical axis of revolution

Example 1: Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x-axis ($0 \leq x \leq \pi$) about the x-axis

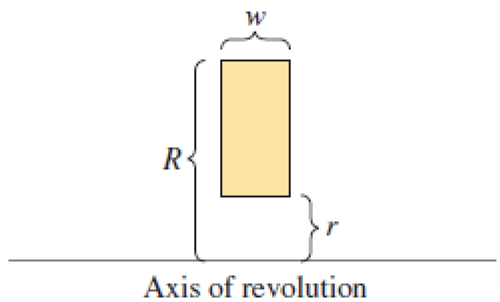
Example 2: Using a Line That Is Not a Coordinate Axis

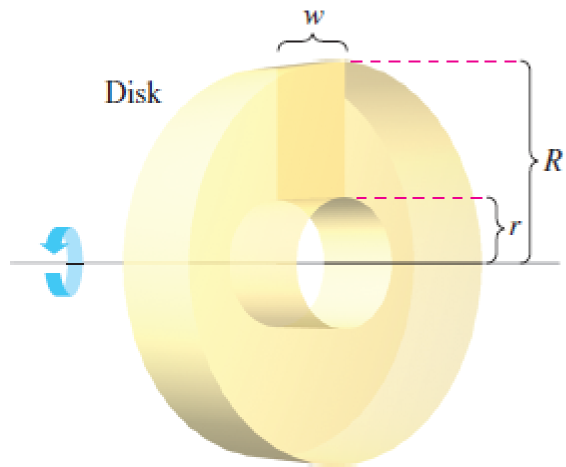
Find the volume of the solid formed by revolving the region bounded by the graphs of

$$f(x) = 2 - x^2$$

and $g(x) = 1$ about the line $y = 1$.

The Washer Method

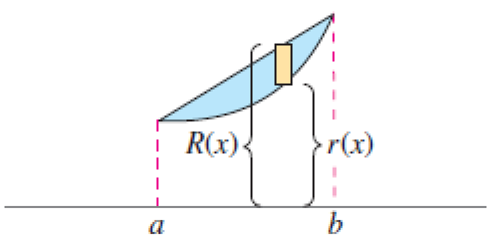




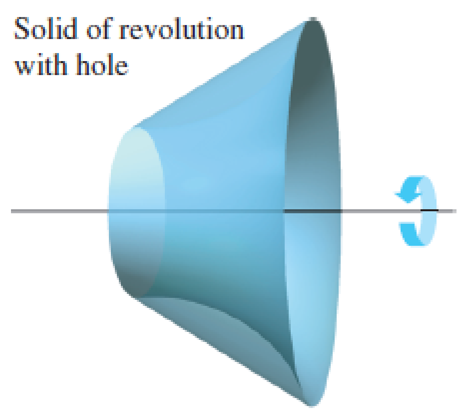
Solid of revolution

$$\text{Volume of washer} = \pi(R^2 - r^2)w$$

Washer Method



Plane region



$$V = \pi \int_a^b [(R[x])^2 - (r[x])^2)dx$$

Example 3: Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = \sqrt{x} \qquad \text{and} \qquad y = x^2$$

about the x-axis.

Example 4: Integrating with Respect to `y`: Two-Integral Case

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

about the y-axis

Solids with Known Cross Sections

Example 1 | Example 2

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$V = \int_a^b A(x)dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$V = \int_c^d A(y)dy$$

Example 6: Triangular Cross Sections

The base of a solid is the region bounded by the lines

$$f(x) = 1 - \frac{x}{2}, \quad g(x) = -1 + \frac{x}{2} \quad \text{and} \quad x = 0.$$

The cross sections perpendicular to the x-axis are equilateral triangles.

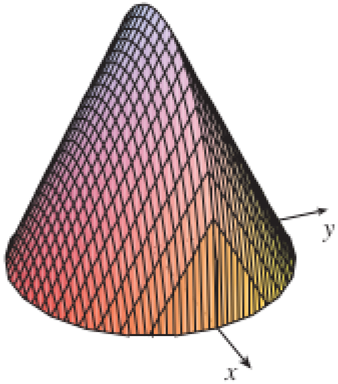
Exercise Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Exercise The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $y = 2$.

Exercise Find the volume of the solid obtained by rotating the region in the previous Example about the line $x = -1$.

Exercise Figure below shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



7.3: Volume: The Shell Method

> **Objectives**
>
> 1. Find the volume of a solid of revolution using the shell method.
> 2. Compare the uses of the disk method and the shell method.

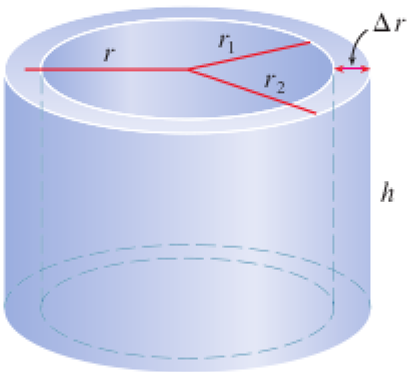
Problem Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the $y-$axis.

Step 1: ☐ Step 2: ☐ Step 3: ☐

""

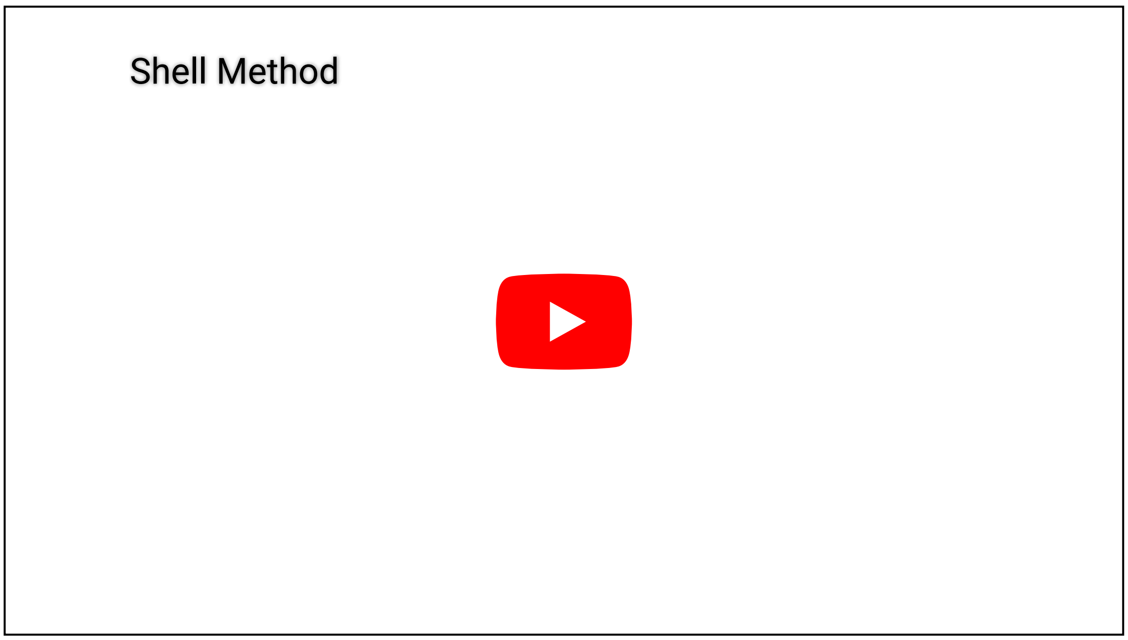
The Shell Method

A shell is a hallow circular cylinder



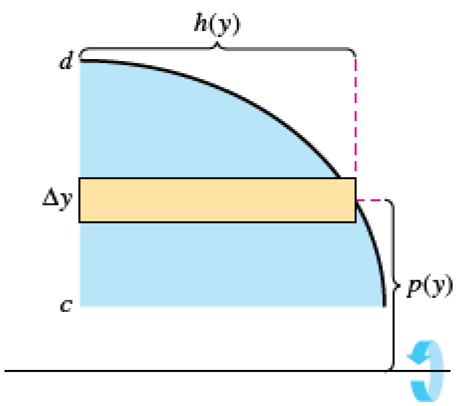
$$V = 2\pi r h \Delta r = [\text{circumference}][\text{height}][\text{thickness}]$$

Cylindrical Shells Illustration



Horizontal Axis of Revolution

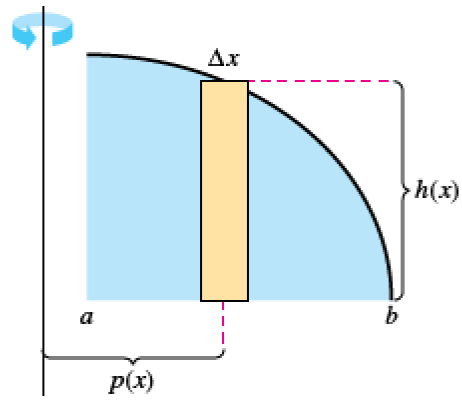
$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)dy$$



Horizontal axis of revolution

Vertical Axis of Revolution

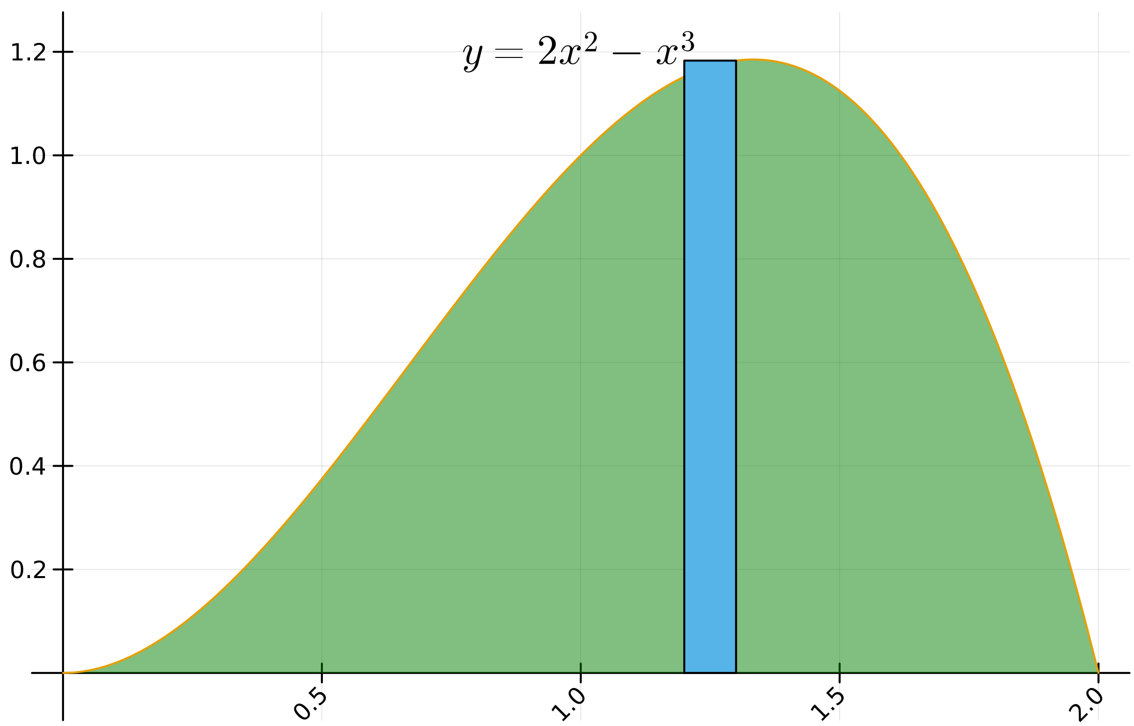
$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)dx$$



Vertical axis of revolution

Example: Find the volume of the solid generated by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y—axis.

Solution:



Example : Find the volume of the solid obtained by rotating about the y—axis the region between $y = x$ and $y = x^2$.

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Example 4: **Shell Method Preferable**

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1.$$

about the y-axis.

Example 5: **Shell Method Necessary**

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$.

7.4: Arc Length and Surfaces of Revolution

> **Objectives**
>
> 1. Find the arc length of a smooth curve.
> 2. Find the area of a surface of revolution.

Arc Length

Definition **Arc Length**

Let the function $y = f(x)$ represents a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 2: **Finding Arc Length**

Find the arc length of the graph of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$.

Example 3: **Finding Arc Length**

Find the arc length of the graph of $(y - 1)^3 = x^2$ on the interval $[0, 8]$.

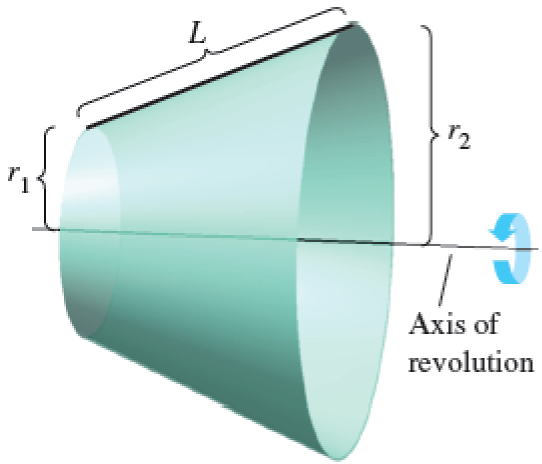
Example 4: **Finding Arc Length**

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$.

Area of a Surface of Revolution

Definition **Surface of Revolution**

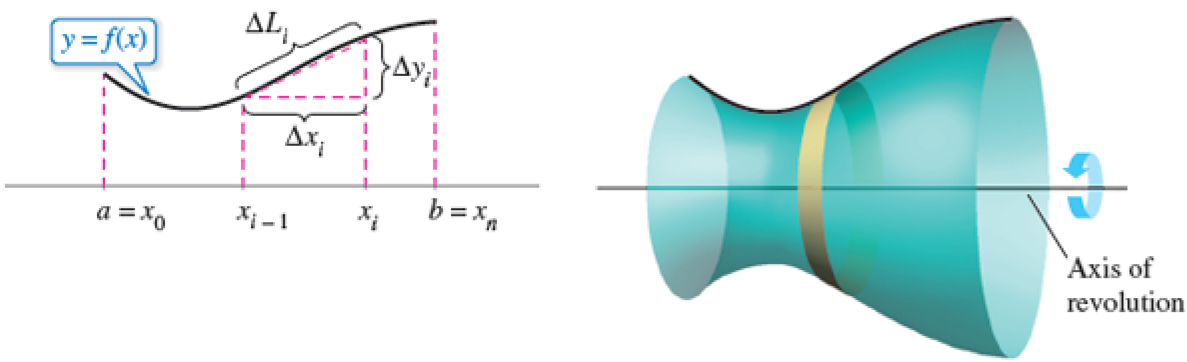
When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.



Surface Area of *frustum*

$$S = 2\pi r L, \quad \text{where} \quad r = \frac{r_1 + r_2}{2}$$

Consider a function f that has a continuous derivative on the interval $[a, b]$. The graph of f is revolved about the x-axis

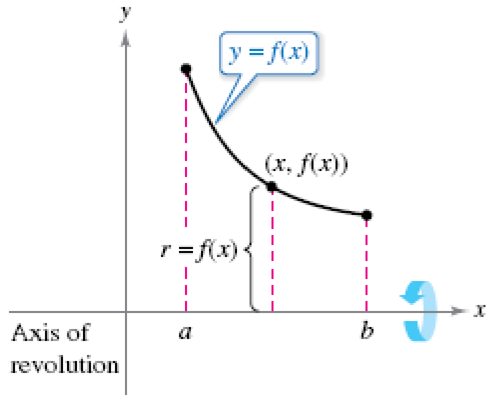


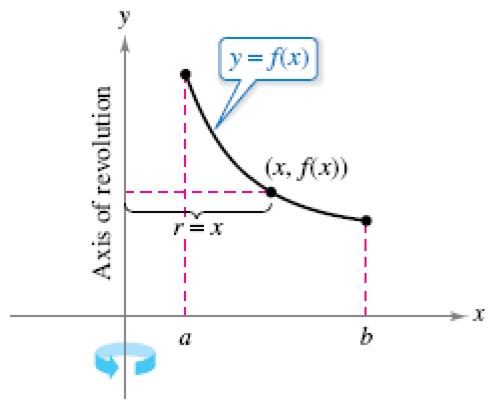
Surface Area Formula

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}dx.$$

Definition **Area of a Surface of Revolution**

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$.





The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx, \quad \textbf{\textit{y} is a function of x .}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution.

If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_a^b r(y)\sqrt{1 + [g'(y)]^2}dy, \quad \textbf{\textit{x} is a function of y .}$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Remark

The formulas can be written as

$$S = 2\pi \int_a^b r(x)ds, \quad y \text{ is a function of x }.$$

and

$$S = 2\pi \int_c^d r(y)ds, \quad x \text{ is a function of y }.$$

where

$$ds = \sqrt{1 + \left[f'(x)\right]^2}dx \quad \text{and} \quad ds = \sqrt{1 + \left[g'(y)\right]^2}dy \quad \text{respectively.}$$

Example 6: **The Area of a Surface of Revolution**

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis.

Example 7: **The Area of a Surface of Revolution**

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis.

8.1: Basic Integration Rules

> **Objectives**
>
> 1. Review procedures for fitting an integrand to one of the basic integration rules.

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

5. $\int \frac{du}{u} = \ln|u| + C$

6. $\int e^u du = e^u + C$

7. $\int a^u\,du = \left(\frac{1}{\ln a}\right) a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \frac{du}{\text{[illegible]}} = \arcsin \frac{u}{\text{[illegible]}} + C$

$\int \quad \sqrt{a^2-u^2} \qquad\qquad a$

19. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

20. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Example 3:

Find $\int \frac{x^2}{\sqrt{16-x^6}}dx$.

Example 4: **A Disguised Form of the Log Rule**

Find $\int \frac{dx}{1+e^x}$.

8.2: Integration by Parts

> **Objectives**
>
> 1. Find an antiderivative using integration by parts.

The integration rule that corresponds to the Product Rule for differentiation is called ***integration by parts***

Indefinite Integrals

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Theorem **Integration by Parts**

If u and v are functions of x and have continuous derivatives, then

$$\int udv = uv - \int vdu.$$

Example 1: Integration by Parts

Find $\int xe^x dx$.

Example 2: Integration by Parts

Find $\int x^2 \ln x dx$.

Example 3: An Integrand with a Single Term

Evaluate $\int_0^1 \arcsin x dx$.

Example 4: Repeated Use of Integration by Parts

Find $\int x^2 \sin x dx$.

Example 5: Integration by Parts

Find $\int \sec^3 x dx$.s

Example 7: Using the Tabular Method

Find $\int x^2 \sin 4x dx$.

8.3: Trigonometric Integrals

> **Objectives**
>
> 1. Solve trigonometric integrals involving powers of sine and cosine.
> 2. Solve trigonometric integrals involving powers of secant and tangent.
> 3. Solve trigonometric integrals involving sine-cosine products.

RECALL

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\begin{aligned} \sin mx \sin nx &= \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x], \\ \sin mx \cos nx &= \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)], \\ \cos mx \cos nx &= \tfrac{1}{2}[\cos(m-n) + \cos(m+n)], \end{aligned}$$

$$\int \tan x dx = \ln|\sec x| + C, \quad \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x dx = -\ln|\csc x| + C, \quad \int \csc x dx = \ln|\csc x - \cot x| + C$$

Integrals of Powers of Sine and Cosine

$$\int \sin^m x \cos^n x dx$$

- m is odd, write as $\int \sin^{m-1} x \cos^n x \sin x dx$. *Example*: $\int \sin^5 x \cos^2 x dx$
- n is odd, write as $\int \sin^m x \cos^{n-1} \cos x dx$. *Example* $\int \sin^5 x \cos^3 x dx$
- m and n are even, use formulae (*Example* $\int \cos^2 x dx$ and $\int \sin^4 x dx$)

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}.$$

Example 1: **Power of Sine Is Odd and Positive**

Find $\int \sin^3 x \cos^4 x dx$.

Example 2: **Power of Cosine Is Odd and Positive**

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$.

Example 3: **Power of Cosine Is Even and Nonnegative**

Find $\int \cos^4 x dx$.

Integrals of Powers of Secant and Tangent

$$\int \tan^m x \sec^n x dx$$

- n is even, write as $\int \tan^m x \sec^{n-2} \sec^2 x dx$. *Example* $\int \tan^6 x \sec^4 x dx$
- m is odd, write as $\int \tan^{m-1} x \sec^{n-1} \tan x \sec x dx$. *Example* $\int \tan^5 x \sec^7 x dx$.

Example 4: **Power of Tangent Is Odd and Positive**

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$.

Example 5: **Power of Secant Is Even and Positive**

Find $\int \sec^4 3x \tan^3 3x dx$.

Example 6: Power of Tangent Is Even

Evaluate $\int_0^{\pi/4} \tan^4 x dx$.

Example 7: Converting to Sines and Cosines

Find $\int \frac{\sec x}{\sqrt{\tan^2 x}} dx$.

Integrals Involving Sine-Cosine Products

Example 8: Using a Product-to-Sum Formula

Find $\int \sin 5x \cos 4x dx$.

8.4: Trigonometric Substitution

> **Objectives**
>
> 1. Use trigonometric substitution to find an integral.
> 2. Use integrals to model and solve real-life applications.

Trigonometric Substitution

We use **trigonometric substitution** to find integrals involving the radicals

$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \sqrt{u^2 - a^2}.$$

Example 1: **Trigonometric Substitution**

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Example 2: **Trigonometric Substitution**

Find $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$.

Example 3: **Trigonometric Substitution: Rational Powers**

Find $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$.

Example 4: **Converting the Limits of Integration**

Find $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Applications

Example 5: **Finding Arc Length**

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

8.5: Partial Fractions

> **Objectives**
>
> 1. Understand the concept of partial fraction decomposition.
> 2. Use partial fraction decomposition with linear factors to integrate rational functions.
> 3. Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions

We learn how to integrate rational function: quotient of polunomial.

$$f(x) = \frac{P(x)}{Q(x)}, \qquad P, Q \text{ are polynomials}$$

How?

- **STEP 0** : if degree of P is greater than or equal to degree of Q goto **STEP 1**, else GOTO **STEP 2**.
- **STEP 1** : Peform long division of P by Q to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

and apply **STEP 2** on $\frac{R(x)}{Q(x)}$.

- **STEP 2** : Write the **partial fractions decomposition**
- **STEP 3** : Integrate

Partial Fractions Decomposition

We need to write $\frac{R(x)}{Q(x)}$ as sum of **partial fractions** by **factor** $Q(x)$. Based on the factors, we write the decomposition accoding to the following cases

case 1: $Q(x)$ is a product of distinct linear factors. we write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

then there exist constants $A_1, A_2, \cdots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

case 2: $Q(x)$ is a product of linear factors, some of which are repeated. say first one

$$Q(x) = (a_1x + b_1)^r(a_2x + b_2)\cdots(a_kx + b_k)$$

then there exist constants $B_1, B_2, \cdots B_r, A_2, \cdots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \left[\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \cdots \frac{B_r}{(a_1x + b_1)^r}\right] + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated. say we have (Note: the quadratic factor $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$). For eaxmple if

$$Q(x) = (ax^2 + bx + c)(a_1x + b_1)$$

then there exist constants A, B, and C such that

$$\frac{R(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{C}{a_1x + b_1}$$

case 4: $Q(x)$ contains irreducible quadratic factors, some of which are repeated. For example if

$$Q(x) = (ax^2 + bx + c)^r(a_1x + b_1)$$

then there exist constants $A_1, B_1, A_2, B_2, \cdots A_r, B_r$ and C such that

$$\frac{R(x)}{Q(x)} = \left[\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}\right] + \frac{C}{a_1x + b_1}$$

Example: Partial Fractions

Write out the form of the partial fractions decomposition of the function

$$\frac{x^3+x+1}{x(x-1)(x+1)^2(x^2+x+1)(x^2+4)^2}$$

More Examples

Find

$$
\begin{aligned}
&(1) \quad \int \frac{1}{x^2-5x+6}dx. \\
&(2) \quad \int \frac{5x^2+20x+6}{x^3+2x^2+x}dx. \\
&(3) \quad \int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx. \\
&(4) \quad \int \frac{8x^3+13x}{(x^2+2)^2}dx. \\
&(5) \quad \int \frac{x^3+x}{x-1}dx. \\
&(6) \quad \int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx. \\
&(7) \quad \int \frac{dx}{x^2-a^2}, \text{ where } a \neq 0 \\
&(8) \quad \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx \\
&(9) \quad \int \frac{2x^2-x+4}{x^3+4x}dx \\
&(10) \quad \int \frac{4x^2-3x+2}{4x^2-4x+3}dx \\
&(11) \quad \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx
\end{aligned}
$$

Remarks

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Rationalizing Substitutions Find

$$\begin{aligned} &(1) \quad \int \frac{\sqrt{x+4}}{x} dx. \\ &(2) \quad \int \frac{dx}{2\sqrt{x+3}+ x}. \end{aligned}$$

8.7: Rational Functions of Sine & Cosine

Special Substitution ($u = \tan\left(\frac{x}{2}\right), \quad -\pi < x < \pi$) **(for rational functions of $\sin x$ and $\cos x$)**

$$dx = \frac{2}{1+u^2} du, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad \sin x = \frac{2u}{1+u^2}$$

$$\begin{aligned} &(1) \quad \int \frac{dx}{3\sin x - 4\cos x}. \\ &(2) \quad \int_0^{\frac{\pi}{2}} \frac{\sin 2x \; dx}{2+\cos x}. \end{aligned}$$

8.8: Improper Integrals

> **Objectives**
>
> - Evaluate an improper integral that has an infinite limit of integration.
> - Evaluate an improper integral that has an infinite discontinuity.

Do you know how to evaluate the following?

$$(1) \quad \int_1^\infty \frac{1}{x^2} dx \quad \text{(Type 1)}$$

$$(2) \quad \int_0^2 \frac{1}{x-1} dx \quad \text{(Type 2)}$$

Improper Integrals with Infinite Limits of Integration

Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called ***convergent*** if the corresponding limit exists and ***divergent*** if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

In part (c) any real number can be used

Example: Determine whether the following integrals are convergent or divergent.

$$(1) \quad \int_1^{\infty} \frac{1}{x^2} dx$$

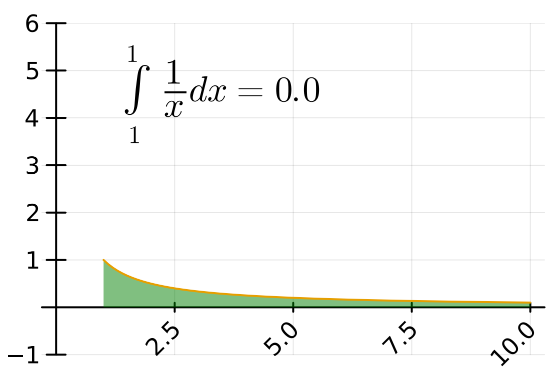
$$(2) \quad \int_1^{\infty} \frac{1}{x} dx$$

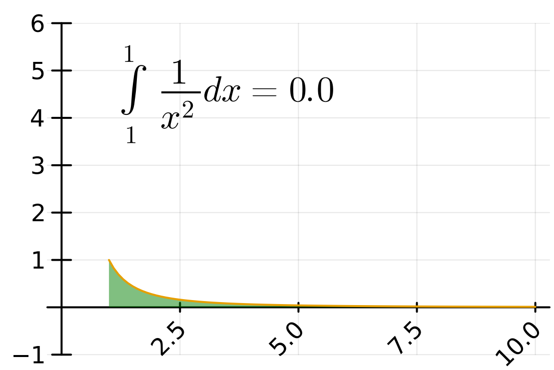
$$(3) \quad \int_0^{\infty} e^{-x} dx$$

$$(4) \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

---

t = 1





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p = 1.0

$$\int_{1}^{\infty} \frac{1}{x} dx = oo$$

Remark

$$\int_{1}^{\infty} \frac{1}{x^p} dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

Improper Integrals with Infinite Discontinuities

Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example:

$$(1) \quad \int_2^5 \frac{1}{\sqrt{x-2}} dx$$

$$(2) \quad \int_0^3 \frac{1}{1-x} dx$$

$$(3) \quad \int_0^1 \ln x dx$$

Example 9: **Doubly Improper Integral**

Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

9.1: Sequences

> Objectives
>
> - Write the terms of a sequence.
> - Determine whether a sequence converges or diverges.
> - Write a formula for the th term of a sequence.
> - Use properties of monotonic sequences and bounded sequences.

Sequence: A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

- a_1: first term,
- a_2: second term,
- a_3: third term,
- $\vdots$
- a_n: n^{th} term,

For example:

- $1, 2, 3, \cdots$
- $1, 1/2, 1/3, \cdots$
- $-1, 1, -1, \cdots$

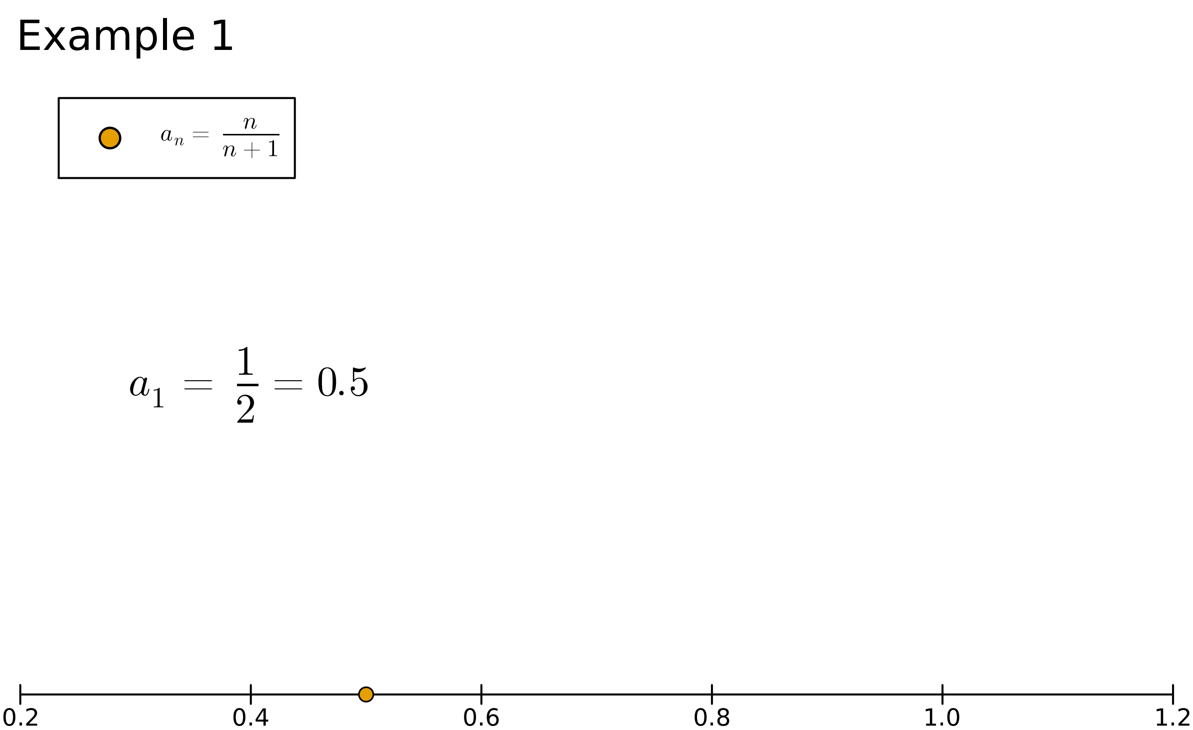
Notation:

- $\{a_1, a_2, a_3, \cdots, a_n, \cdots\} = \{a_n\}$ or
- $\{a_1, a_2, a_3, \cdots, a_n, \cdots\} = \{a_n\}_{n=1}^{\infty}$

More examples

- $\left\{\frac{n}{n+1}\right\}$
- $\left\{\frac{(-1)^n(n+1)}{5^n}\right\}$
- $\left\{\sqrt{n-4}\right\}_{n=4}^{\infty}$
- $a_1 = 1, a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ (**Fibonacci sequence**)

n = 1

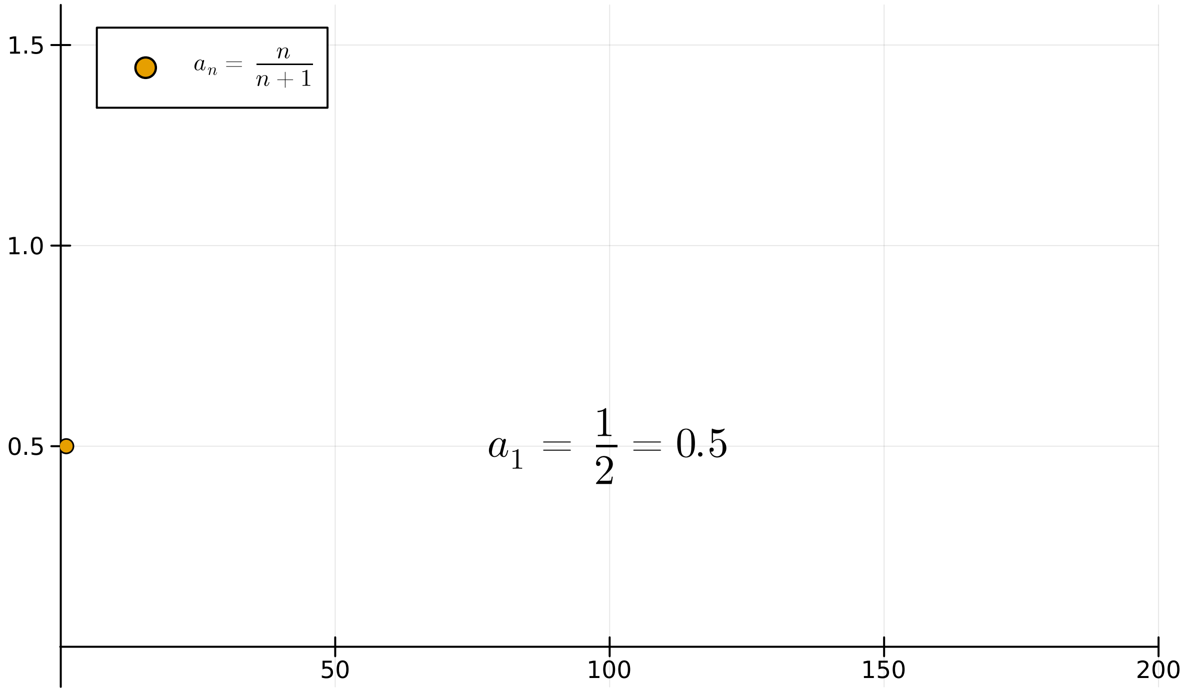


Visualization

1. On a number line (as above)
2. By plotting graph

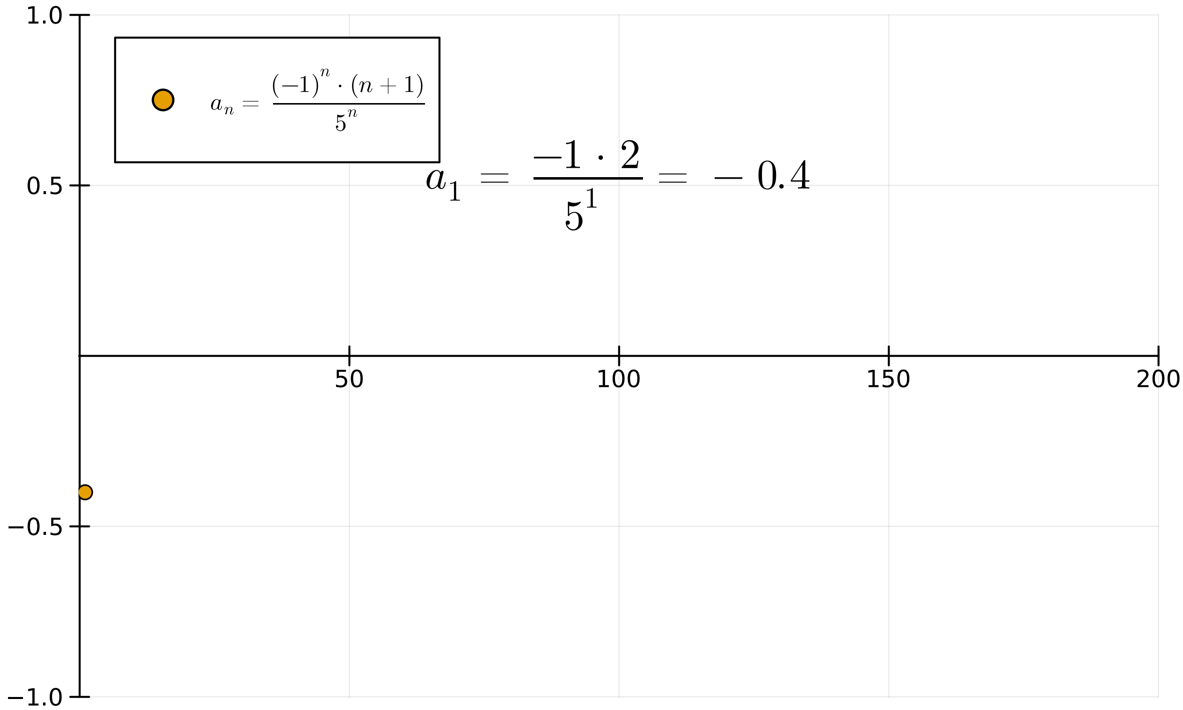
n = 1

Example 1 (Graph)

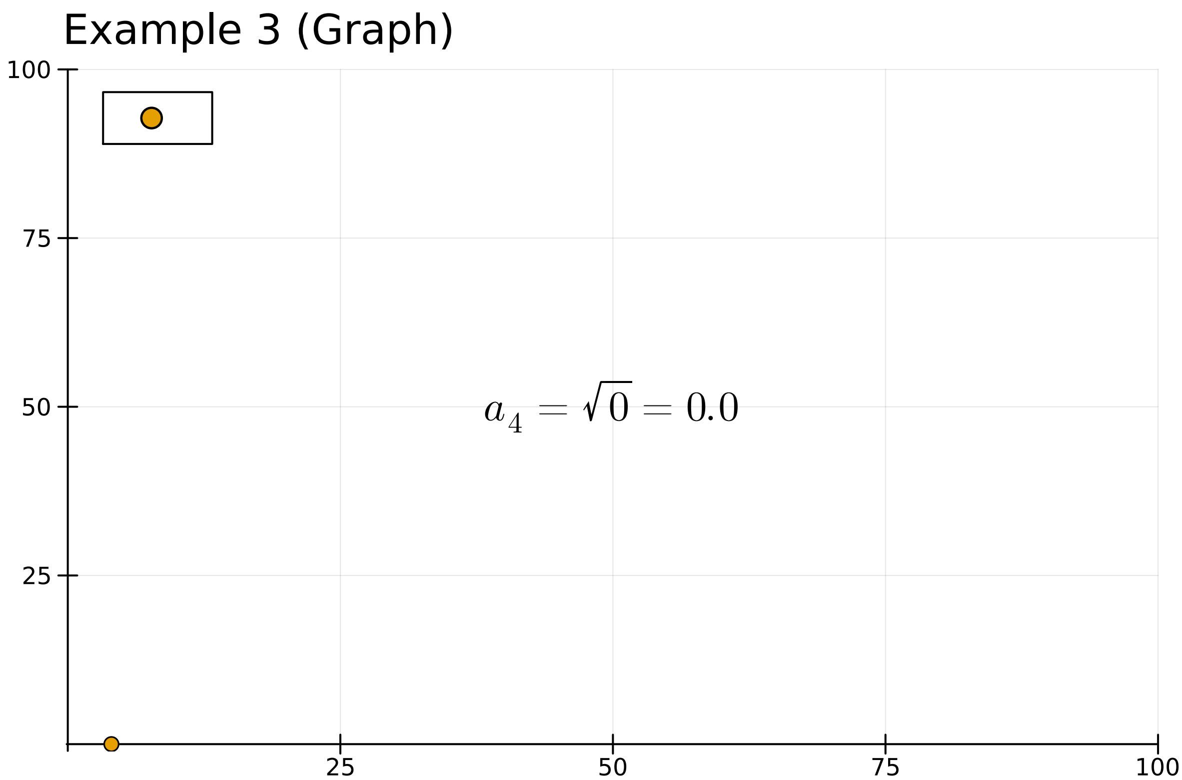


n = 1

Example 2 (Graph)



n = 4



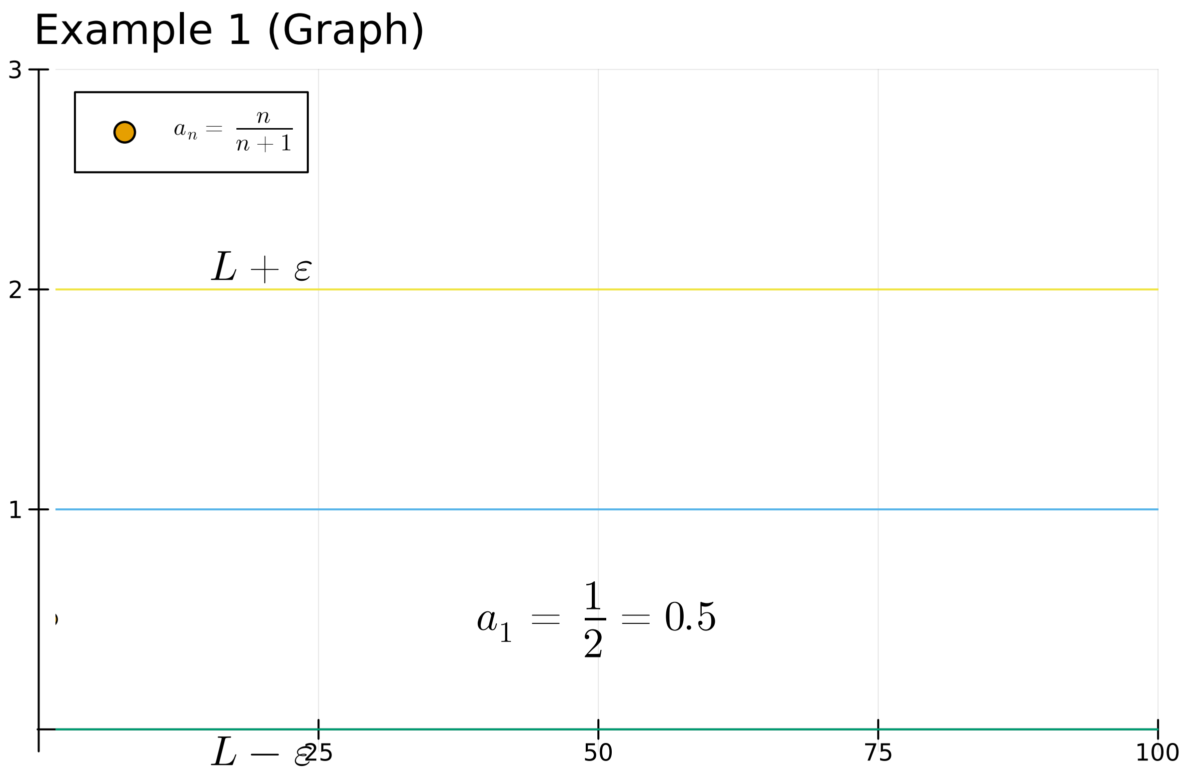
What are trying to study?

- **convergence** (what happended when n gets larger and larger $n \to \infty$)

For ***Example 1***: $a_n = \frac{n}{n+1}$, it is fair to say and write

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

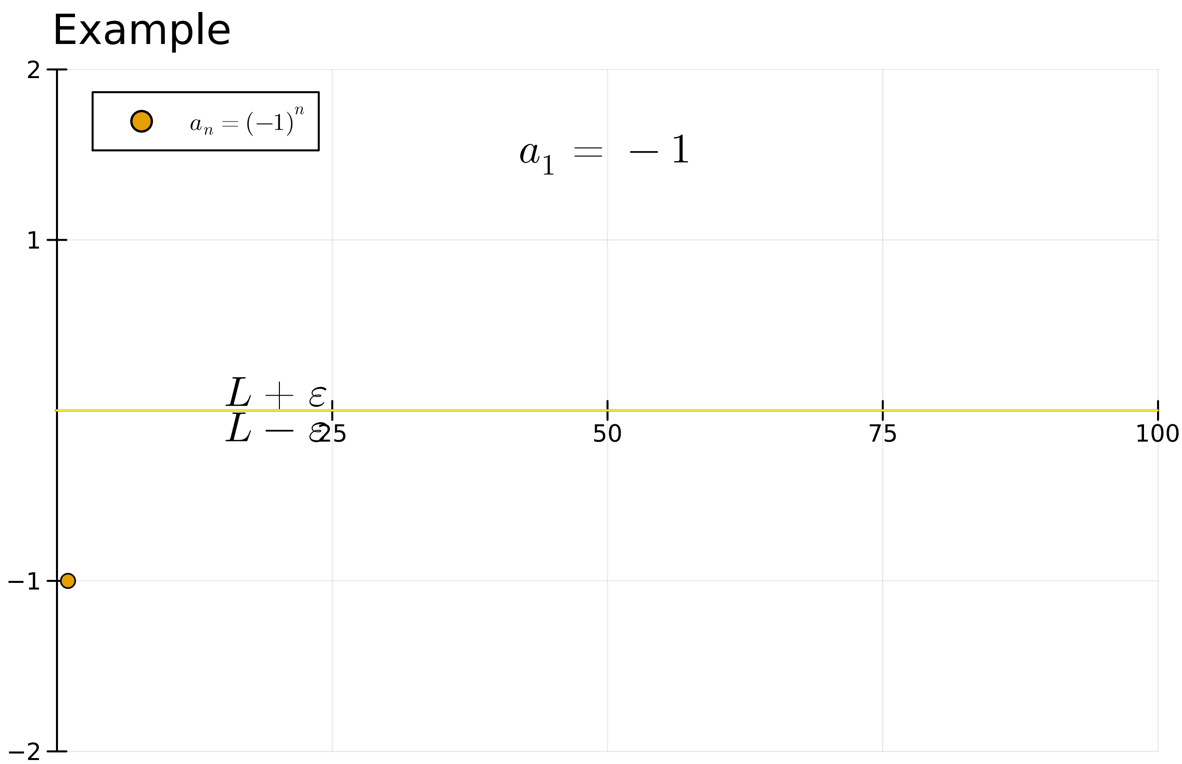
ϵ = 1.0   n = 1



Example

$$\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \cdots\}$$

ϵ =	0.0	n =	1
$L =$	0.0		



Remark:

$$\lim_{n\to\infty} (-1)^n \quad \text{DNE}$$

Limit of a Sequence

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\} is$L``, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\epsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \epsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges**.

Theorem **Limit of a Sequence**

If

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad f(n) = a_n \quad \text{when } n \text{ is an integer,}$$

then

$$\lim_{n\to\infty} a_n = L.$$

- **Remark**

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if} \quad r > 0$$

Limit Laws for Sequences Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. **Sum Law**

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

2. **Difference Law**

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

3. **Constant Multiple Law**

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

4. **Product Law**

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

5. **Quotient Law**

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

Power Law

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$$

Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L.$$

Theorem

If

$$\lim_{n\to\infty} |a_n| = 0,$$

then

$$\lim_{n\to\infty} a_n = 0.$$

Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Remark

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1, \\ 1 & \text{if} \quad r = 1 \end{cases}$$

Examples

Find

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
2. $\lim_{n\to\infty}\frac{n^2}{2^n-1}$.
3. $\lim_{n\to\infty}\frac{n}{n+1}$.
4. $\lim_{n\to\infty}\frac{n}{\sqrt{n+1}}$.
5. $\lim_{n\to\infty}\frac{\ln n}{n}$.
6. $\lim_{n\to\infty}\frac{(-1)^n}{n}$.
7. $\lim_{n\to\infty}\sin\left(\pi/n\right)$.
8. $\lim_{n\to\infty}\frac{n!}{n^n}$.
9. $\lim_{n\to\infty}\frac{(-1)^n}{n!}$.

Exercise

$$\lim_{n\to\infty}\frac{n^n}{n!}.$$

Pattern Recognition for Sequences

Example

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1},\frac{4}{3},\frac{8}{5},\frac{16}{7},\frac{32}{9},\ldots$$

and then determine whether the sequence you have chosen converges or diverges.

Example

Find a sequence $\{a_n\}$ whose first five terms are

$$-\frac{2}{1},\frac{8}{2},-\frac{26}{6},\frac{80}{24},-\frac{242}{120},\ldots$$

and then determine whether the sequence you have chosen converges or diverges.

Monotonic and Bounded Sequences

Definition

- A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is,$a_1 < a_2 < a_3 < \cdots$.
- It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
- A sequence is called **monotonic** if it is either increasing or decreasing.

Examples

Is the following increasing or decreasing?

1. $\left\{\frac{3}{n+5}\right\}$.
2. $\left\{\frac{n}{n^2+1}\right\}$.

Definition

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

A sequence is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

Example

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6), \quad \text{for } n = 1, 2, 3, \cdots$$

9.2: Series and Convergence

> **Objectives**
>
> - Understand the definition of a convergent infinite series.
> - Use properties of infinite geometric series.
> - Use the th-Term Test for Divergence of an infinite series.

Infinite Series

Consider the sequence $\{a_n\}_{n=1}^{\infty}$. The expression

$$a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series** (or simply **series**) and we use the notation

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

To make sense of this sum, we define a related **sequence** called the sequence of **partial sums** $\{s_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \textstyle\sum_{i=1}^{n} a_i \\ &\vdots \end{aligned}$$

and give the following definition

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $s_n$denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the $\sum a_n$series is called **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Remark

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Exercise Assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence.

1. Find

$$\sum_{n=1}^{\infty} a_n \quad \text{if} \quad s_n = \sum_{i=1}^{n} a_i = \frac{n+2}{3n-5}$$

2. Can you find a_n?

Solution

1. We find first

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n+2}{3n-5} = \frac{1}{3}$$

since the sequence $\{s_n\}$ converges to $\frac{1}{3}$, then the series converges and its sum is

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3}$$

2. Note that

$$\begin{aligned} a_n &= s_n - s_{n-1} = \frac{n+2}{3n-5} - \frac{(n-1)+2}{3(n-1)-5} \\ &= \frac{n+2}{3n-5} - \frac{n+1}{3n-8} \\ &= \frac{(n+2)(3n-8)-(n+1)(3n-5)}{(3n-5)(3n-8)} \end{aligned}$$

so,

$$a_n = \frac{-11}{(3n-5)(3n-8)}$$

Telescoping sum

Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution in class

Recall

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad |r| < 1(-1 < r < 1), \\ 1 & \text{if} \quad r = 1, \end{cases}$$

So $\{r^n\}$ converges if $r \in (-1, 1]$ and diverges otherwise

Geometric Series

The series

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

is called the **geometric series** with **common ration** r

It is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1$$

and divergent if $|r| \geq 1$.

Remark In words: the sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

Examples

1. Find the sum of the geomtric series

$$4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$$

2. Is the series

$$\sum_{n=1}^{\infty} 2^{2n}\, 3^{1-n} \quad \text{convergent or divergent?}$$

3. Write $2.\overline{7}$ as rational number (ratio of integers).
4. Find the sum of the series

$$\sum_{n=0}^{\infty} x^n \quad \text{where} \quad |x| < 1.$$

Test for Divergence

Example Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n \to \infty} a_n = 0.$$

Proof

$$a_n = s_n - s_{n-1}$$

Divergence Test

$$\text{If } \lim_{n \to \infty} a_n \neq 0 \quad \text{or} \quad \lim_{n \to \infty} a_n \text{ DNE} \quad \text{then the series} \quad \sum_{n=1}^{\infty} a_n \text{ is divergenet}$$

Example

$$\text{The series} \quad \sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^2 + 5} \quad \text{is divergent.}$$

Properties of Convergent Series

Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

$$\begin{array}{llll} \text{(i)} & \sum_{n=1}^{\infty} ca_n & = & c\sum_{n=1}^{\infty} a_n \\ \text{(ii)} & \sum_{n=1}^{\infty} (a_n + b_n) & = & \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ \text{(iii)} & \sum_{n=1}^{\infty} (a_n - b_n) & = & \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \end{array}$$

Remark

If it can be shown that

$$\sum_{n=100}^{\infty} a_n$$

is convergent. Then

$$\sum_{n=1}^{\infty} a_n$$

is convergent.

9.3: The Integral Test and p-Series

> **Objectives**
>
> - Use the Integral Test to determine whether an infinite series converges or diverges.
> - Use properties of -series and harmonic series.

The Integral Test and Estimates of Sums

Suppose f a function that is

1. continuous on $[1, \infty)$,
2. positive on $[1, \infty)$,
3. decreasing on $[1, \infty)$

and let $a_n = f(n)$. Then the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent if and only if the improper integral

$$\int_1^{\infty} f(x)dx$$

is convergent. In other words:

1. If $\int_1^{\infty} f(x)dx$ is convergent, then is $\sum_{n=1}^{\infty} a_n$ convergent.
2. If $\int_1^{\infty} f(x)dx$ is divergent, then is $\sum_{n=1}^{\infty} a_n$ divergent.

Examples

Test for convergence

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

Solution in class

Remark

$$\text{The series } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent but } \sum_{n=1}^{\infty} \frac{1}{n^2} \neq 1.$$

$$\text{It sum is actually equal to } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

P-series and the Harmonic Series

$$\text{The } p - \text{series } \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is convergent if } p > 1 \text{ and is divergent if } p \leq 1.$$

---

1. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ is divergent; because it is a $p-$series with $p = \frac{1}{3} < 1$.
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent; because it is a $p-$series with $p = 3 > 1$.

Example

Show that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

is divergent.

Estimating the Sum of a Series

Suppose that the **integral test** is used to show that

$$\sum_{n=1}^{\infty} a_n$$

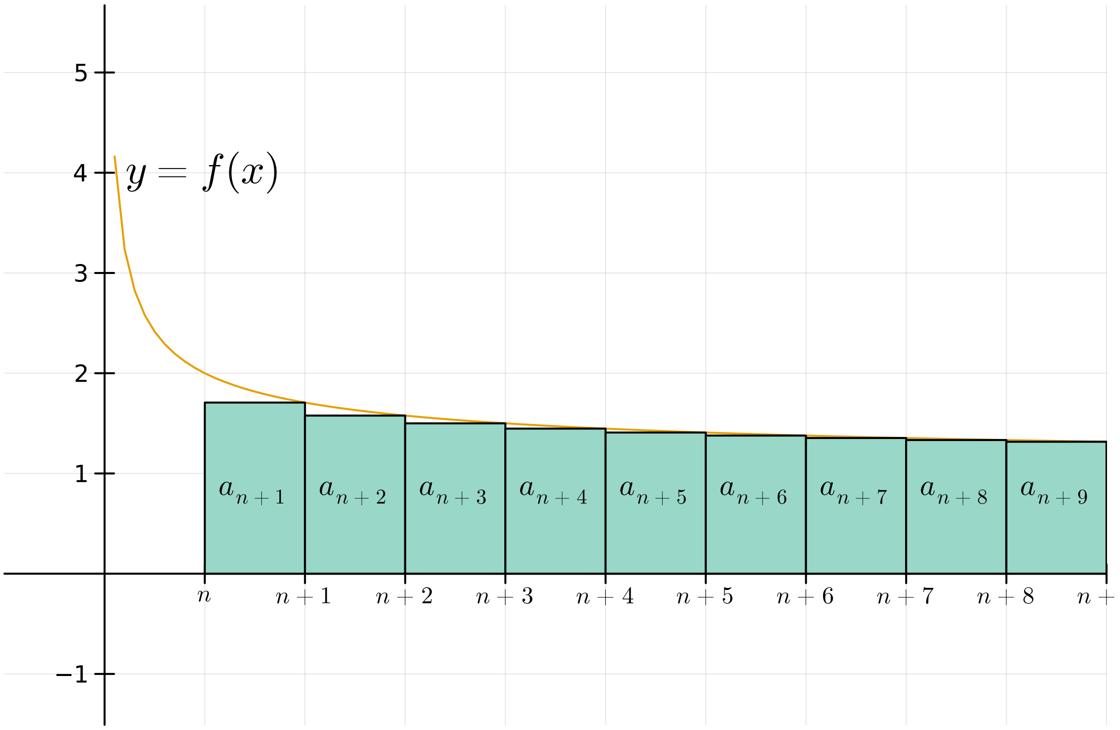
is **convergent**. So its sequenc of partial sums $\left\{s_n = \sum_{i=1}^{n} a_i\right\}$ is convergent; that is

$$\lim_{n\to\infty} s_n = s.$$

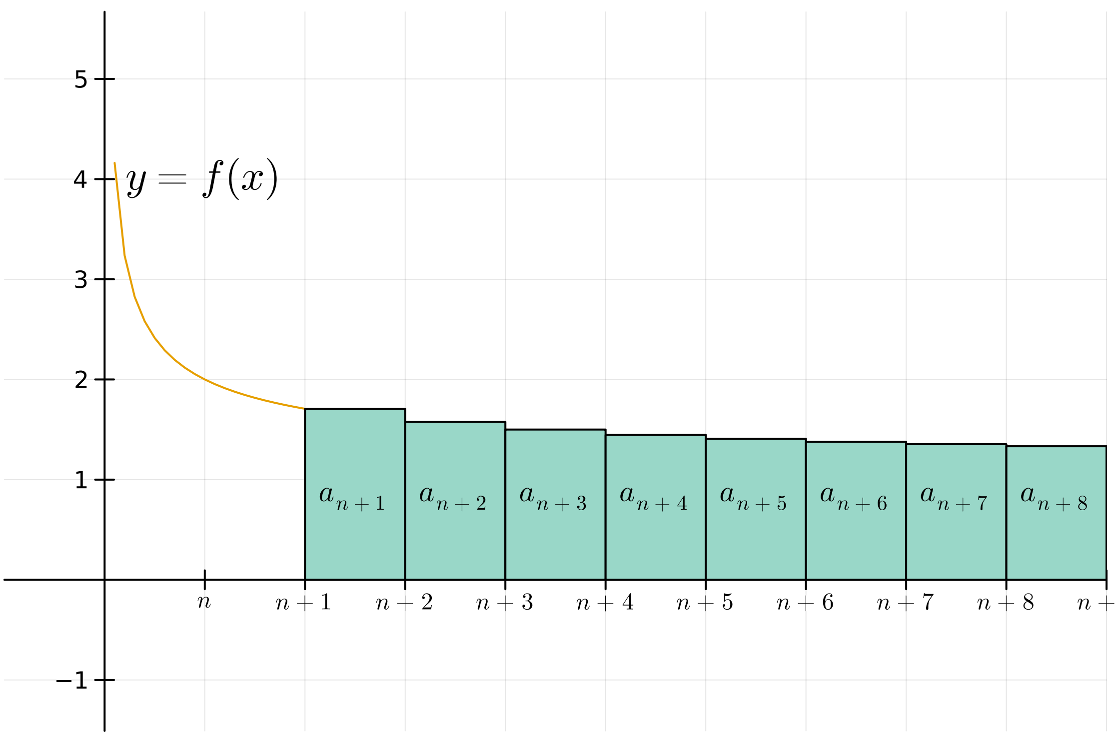
So we can write

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{s} = \underbrace{\sum_{i=1}^{n} a_i}_{s_n} + \underbrace{\sum_{i=n+1}^{\infty} a_i}_{R_n}$$

R_n is the **Remainder** or the error when s_n is used to approximate s.



$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^{\infty} f(x)dx$$



$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)dx$$

```
(0.005, 0.00413223)
```

```
1/200, 1/242
```

Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

9.4: Comparisons of Series

> **Objectives**
>
> - Use the Direct Comparison Test to determine whether a series converges or diverges.
> - Use the Limit Comparison Test to determine whether a series converges or diverges.

The Comparison Tests

The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then is $\sum a_n$ also divergent

---

Remarks

- Most of the time we use one of these series:
  - $p-$series $\sum \frac{1}{n^p}$
  - geometric series.

Examples Test for convegence

$$(1) \quad \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

The Limit Comparison Test

Suppose $\sum a_n$ that and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Remark

Exercises `40` and `41` deal with the cases $c = 0$ and $c = \infty$.

Examples Test for convegence

$$(3) \quad \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$(4) \quad \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

Exercises

Test for convegence

$$(5) \quad \sum_{n=1}^{\infty} \frac{n + 3^n}{n + 2^n}$$

$$(6) \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

9.5: Alternating Series

> **objectives**
>
> - Use the Alternating Series Test to determine whether an infinite series converges.
> - Use the Alternating Series Remainder to approximate the sum of an alternating series.
> - Classify a convergent series as absolutely or conditionally convergent.
> - Rearrange an infinite series to obtain a different sum.

An **alternating series** is a series whose terms are alternately **positive** and **negative**. For examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

$$\text{alternating series}\quad \sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots\qquad(b_n>0)$$

satisfies the conditions

$$\text{(i)}\quad b_{n+1}\le b_n\qquad\text{for all } n$$

$$\text{(ii)}\quad \lim_{n\to\infty}b_n=0$$

then the series is convergent.

n = 1

Proof

$$s_1 = b_1$$

Example Test for convegrnce

$$(1) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$$

$$(3) \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

Estimating Sums of Alternating Series

If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$\text{(i)} \quad b_{n+1} \le b_n \quad \text{and}$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \le b_{n+1}$$

Example How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$$

do we need to add in order to find the sum accurate with $|error| < 0.000001$?

Absolute Convergence and Conditional Convergence

- A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, $\sum a_n$ if converges but $\sum |a_n|$ diverges.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Examples Determine whether the series is absolutely convergent, conditionally convergent, or divergent

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

$$\text{(v)} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$$

9.6: The Ratio and Root Tests

> **Objectives**
>
> - Use the Ratio Test to determine whether a series converges or diverges.
> - Use the Root Test to determine whether a series converges or diverges.
> - Review the tests for convergence and divergence of an infinite series.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1,$ then the series is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty,$ then the series is divergent

(iii) $If \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$ the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

The Root Test

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1,$ then the series is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty,$ then the series is divergent

(iii) $If \lim_{n\to\infty}\sqrt[n]{|a_n|} = 1,$ the Ratio Test is inconclusive.

Examples Test for convergence

$$(1) \qquad \sum_{n=1}^{\infty} \frac{\cos n}{n^3}$$

$$(2) \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

$$(3) \qquad \sum_{n=1}^{\infty} \left(\frac{2n+5}{5n+2}\right)^n$$

```
[-1.0, 1.0, -1.0, 1.0, -1.0, 1.0, -1.0, 1.0, -1.0, 1.0]
```

```
[cos(π*n) for n in 1:10]
```

9.7: Taylor Polynomials and Approximations

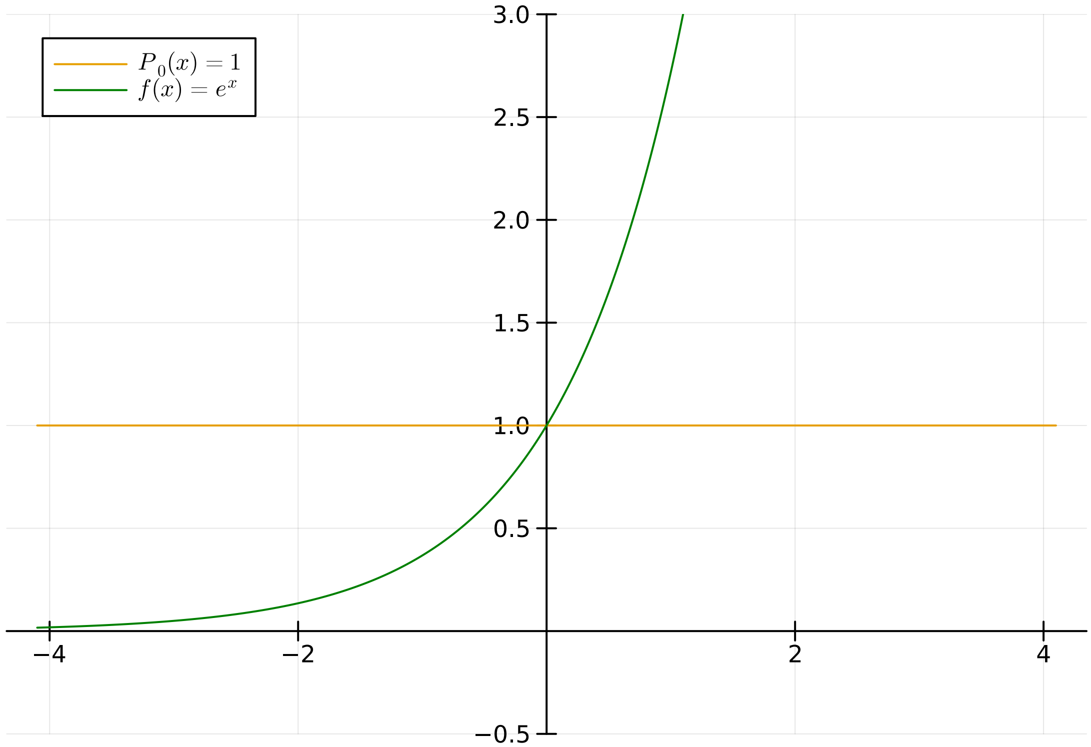
> **Objectives**
>
> - Find polynomial approximations of elementary functions and compare them with the elementary functions.
> - Find Taylor and Maclaurin polynomial approximations of elementary functions.
> - Use the remainder of a Taylor polynomial.

Polynomial Approximations of Elementary Functions

For the function $f(x) = e^x$, find a first-degree polynomial function $P_1(x) = a_0 + a_1 x$ whose value and slope agree with the value and slope of at $x = 0$.

- (•) Show
- ( ) Hide

n = 0



Taylor and Maclaurin Polynomials

Definitions **of th Taylor Polynomial and th Maclaurin Polynomial**

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth **Taylor polynomial** for f at c. If c, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called nth th **Maclaurin polynomial** for f.

Example 3: A Maclaurin Polynomial for e^x

Example 4: Finding Taylor Polynomials for lnx

Find the Taylor polynomials P_0, P_1, P_2, P_3, and P_4 for

$$f(x) = \ln x$$

centered at $c = 1$.

Example 5: Finding Maclaurin Polynomials for cosx

Find the Taylor polynomials P_0, P_2, P_4, and P_6 to approximate $\cos(0.1)$.

Example 6: Finding Taylor Polynomials for sin x

Find the Taylor polynomial $P_3$for

$$f(x) = \sin x$$

centered at $c = \pi/6$.

Example 7: Approximation Using Maclaurin Polynomials

Use a fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

9.8: Power Series

> **Objectives** -Understand the definition of a power series.
>
> - Find the radius and interval of convergence of a power series.
> - Determine the endpoint convergence of a power series.
> - Differentiate and integrate a power series.

A series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is called a **power series in** $(x-c)$ or a **power series centered at** c or a **power series about** c

We are interested in **finding the values of** x **for which this series is convergent.**

Radius and Interval of Convergence

Theorem For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, there are only three possibilities:

(i) The series converges only when $x = c$.

(ii)The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-c| < R$ and diverges if $|x-c| > R$.

Remarks

- The number R is called the **radius of convergence** of the power series.
  - The radius of convergence is $R = 0$ in case (i)
  - $R = \infty$ in case (ii).
- The **interval of convergence** of a power series is the interval that consists of all values of for which the series converges.
  - In case (i) the interval consists of just a single point a.
  - In case (ii) the interval is $(-\infty, \infty)$.

Examples:

$$(1) \quad \sum_{n=0}^{\infty} n! x^n$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

$$(3) \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Endpoint Convergence

Examples

Find the radius of convergence and interval of convergence of the series

$$\begin{array}{ll} (4) & \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}} \\ (5) & \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} \end{array}$$

Differentiation and Integration of Power Series

(term-by-term differentiation and integration)

Theorem

If the power series $\sum a_n(x-c)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$\begin{array}{llll} \text{(i)} & f'(x) & = & a_1 + 2a_2(x-c) + 3a_2(x-c)^2 + \cdots \\ & & = & \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \\ \\ \text{(ii)} & \int f(x)dx & = & C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \\ & & = & C + \sum_{n=0}^{\infty} a_n\frac{(x-c)^{n+1}}{n+1} \end{array}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 8: **Intervals of Convergence**

Consider the function

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

Find the interval of convergence for each of the following.

1. $f(x)$
2. $f'(x)$
3. $\int f(x)dx$

9.9: Representation of Functions by Power Series

> **Objectives**
>
> - Find a geometric power series that represents a function.
> - Construct a power series using series operations.

Geometric Power Series

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n,\quad |x|<1.$$

Examples

1. Express as the sum of a power series and find the interval of convergence.

$$f(x) = \frac{1}{1+x^2}$$

2. Find a power series representation for

$$f(x) = \frac{1}{x+2}$$

3. Find a power series representation for

$$f(x) = \frac{x^3}{x+2}$$

4. Find a power series representation around 1 for

$$f(x) = \frac{1}{x}$$

SOLUTION IN CLASS

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$.
2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{Nn}$.
3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$.

Examples

4. Express as a power series

$$f(x) = \frac{3x-1}{x^2-1}$$

4. Express as a power series

$$f(x) = \frac{1}{(1-x)^2}$$

5. Express as a power series

$$f(x) = \ln(1+x)$$

6. Express as a power series

$$f(x) = \tan^{-1} x$$

7. Evaluate

$$\int \frac{dx}{1+x^7}$$

8. Approximate π

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4} \quad \text{(see ex. 44)}$$

SOLUTION IN CLASS

9.10: Taylor and Maclaurin Series [^⭐]

[^⭐]: Students have to memorize the power series representations of the functions $f(x)=\frac{1}{1+x}, e^x, \sin x, \cos x, \arctan x, (1+x)^k$ in page 674.

> **Objectives**
>
> - Find a Taylor or Maclaurin series for a function.
> - Find a binomial series.
> - Use a basic list of Taylor series to find other Taylor series.

- By the end of this section we will be able to write the following power series representations of certain functions

$$
\begin{array}{llllll}
(1) & \frac{1}{1-x} & = \sum_{n=0}^{\infty} x^n & = 1+x+x^2+x^3+\cdots, & R=1 \\
(2) & \ln(1+x) & = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} & = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots, & R=1 \\
(3) & \tan^{-1} x & = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} & = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots, & R=1 \\
(4) & e^x & = \sum_{n=0}^{\infty} \frac{x^n}{n!} & = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots, & R=\infty \\
(5) & \sin x & = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} & = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots, & R=\infty \\
(6) & \cos x & = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} & = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots, & R=\infty \\
(7) & (1+x)^k & = \sum_{n=0}^{\infty} \binom{k}{n} x^n & = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots, & R=1
\end{array}
$$

Theorem **Taylor Theorem**

If f has a power series representation (expansion) at a , that is, if

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n, \quad |x-c|<R$$

then its coefficients are given by the formula

$$c_n=\frac{f^{(n)}(a)}{n!}$$

Remarks

- The series is called the **Taylor series of the function f at a (or about a or centered at a)**.
- (**Maclaurin Series**) If $a=0$, Taylor series becomes

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Examples (important)

- Find ***Maclaurin*** series for

$$\begin{array}{ll}(1) & f(x)=e^x\\ (2) & f(x)=\sin x\\ (3) & f(x)=\cos x\end{array}$$

- Find Taylor Series of $f(x)=\sin x$ about $\frac{\pi}{3}$.

The Binomial Series

Example: Find the Maclaurin series for $f(x)=(1+x)^k$, where k is any real number.

Solution: In Class

The Binomial Series (Theorem)

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Remarks

- This is called **binomial coefficients**. Note that

$$\binom{k}{n} = 0 \qquad \text{if} \quad k \text{ is integer and } k < n$$

$$\binom{k}{0} = 1, \quad \binom{k}{1} = k$$

- If $-1 < k \leq 0$, it converges at $x = 1$.
- If $k \geq 0$ it converges at $x = \pm 1$.

Example

Find the Maclaurin series for the function

$$f(x) = \frac{1}{\sqrt{4-x}}$$

and its radius of convergence.

Deriving Taylor Series from a Basic List

Check the table

Examples

- Find the Maclaurin series for

$$\begin{aligned} &\text{(a)} \quad f(x) = x\cos x \\ &\text{(b)} \quad f(x) = \ln(1+3x^2) \end{aligned}$$

- Find the function represented by the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n x^n}{n!}$$

- Find the sum of the series

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

More Examples

- Evaluate

$$\int e^{-x^2}\,dx$$

- Evaluate

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

- Find the first 3 nonzero terms of Maclaurin series for

$$(a) \quad e^x \sin x \qquad (b) \quad \tan x$$

- Find the sum of

$$(a) \quad \sum_{n=0}^{\infty} \frac{x^{4n}}{n!} \qquad (b) \quad \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

```julia
begin
    using FileIO, ImageIO, ImageShow, ImageTransformations
    using SymPy
    using PlutoUI
    using CommonMark
    using Plots, PlotThemes, LaTeXStrings
    using HypertextLiteral: @htl, @htl_str
    using Colors
    using Random
end
```